AccuracyOfFiniteContinuedFractionApproximationsForIrrationals

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational, let ξ be its unique associated regular continued fraction of the form $\xi = [b_0; b_1, b_2, \ldots]$, and let $A_n / B_n = [b_0; b_1, b_2, \ldots, b_n]$ denote its nth convergent. The degree of accuracy of the approximation of α by A_n / B_n satisfies

$$\frac{1}{B_n(B_n + B_{n+1})} \leq \left| \alpha - \frac{A_n}{B_n} \right| \leq \frac{1}{B_n B_{n+1}}$$

for all $n \geq 0$.

AdamsMetricalTheorem

Let α and β be irrationals with $0 < \alpha, \beta < 1$,

$$\xi_1 = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{a_n}$$

be the regular continued fraction of α,

$$\xi_2 = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction of β, and let α_n, β_n be the respective continuants. Define

$\psi_\nu(n, \alpha, \beta) =$ number of integers $0 \leq j \leq n - 1$ such that $a_{j+1} = b_1$, $\ldots, a_{j+\nu} = b_\nu$, and $\alpha_{\nu+j+1} > \beta_{\nu+1}$

$\varphi_\nu(n, \alpha, \beta) =$ number of integers $0 \leq j \leq n - \nu$ such that $a_{j+1} = b_1$, $\ldots, a_{j+\nu} = b_\nu$, and $\alpha_{\nu+j+1} > \beta_{\nu+1}$

where ν, n, and j are non-negative integers. Then for almost all α the following identities hold:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{\infty} (-1)^\nu \psi_\nu(n, \alpha, \beta) = \frac{\ln(\beta + 1)}{\ln(2)}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=0}^{\infty} (-1)^\nu \varphi_\nu(n, \alpha, \beta) = \frac{\ln(\beta + 1)}{\ln(2)}.$$

AlgebraicIndependenceCriterionForContinuedFractions

Let $\alpha^{(j)} = \left[b_0^{(j)};\ b_1^{(j)},\ b_2^{(j)},\ \ldots\ \right]$ be regular continued fractions for $j = 1, 2, \ldots, t$ and let their nth convergents be denoted $A_n^{(j)} \big/ B_n^{(j)}$. Then the collection $\alpha^{(1)}, \ldots, \alpha^{(t)}$ is algebraically independent if there exits a bounded function $k : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that (i) $\ln\left(b_{k\,n+n}^t\right) \big/ \ln\left(B_n^1\right)$ is undbounded for all $n \in \mathbb{Z}^+$ and (ii) For $j = 2, 3, \ldots, t$,

$$0 < \liminf_{n\to\infty} \left(\frac{b_n^{(j)}}{b_n^{(j-1)}}\right) < 1 \text{ and } 0 < \limsup_{n\to\infty} \left(\frac{b_n^{(j)} + 1}{b_n^{(j-1)}}\right) < 1.$$

AlgebraicIndependenceOfNNumbers1

Let ξ_i, $i = 1, 2, \ldots, n$ be n irrational numbers with regular continued fraction expansion

$$\xi_i = b_{i,0} + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_{i,j}}$$

with $b_{i,j} \in \mathbb{Z}^+$ and convergents $A_{i,n} \big/ B_{i,n}$.

Let $r > 1$ and let $\left\{n_j\right\}_{j=1}^{\infty}$ be a sequence of increasing positive integers and let $f(n)$ be an integer-valued function for integer argument n and $\lim_{n\to\infty} f(n) = \infty$.

If there exists a subsequence $\{n_i\}_{i=1}^{\infty}$ such that for all $i = 1, 2, \ldots$

$$b_{i,n_j+1} \geq B_{1,n_i}^{f(i)}$$

and

$$B_{j-1,n_i} \geq r^{f(i)}\, B_{j,n_i}$$

and

$$B_{j-1,n_i+1} \geq r^{f(i)}\, B_{j,n_i+1},$$

then the ξ_i are algebraically independent over $\mathbb{Q}$.

AlgebraicIndependenceOfNNumbers2

Let ξ_i, $i = 1, 2, \ldots, n$ be n irrational numbers with regular continued fraction expansion

$$\xi_i = b_{i,0} + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_{i,j}}$$

with $b_{i,j} \in \mathbb{Z}^+$.

Let $r > 1$, $\tau > 1$, and $\{n_j\}_{j=1}^{\infty}$ be a sequence of increasing positive integers and $f(n)$ be an integer-valued function for integer argument n and $\lim_{n\to\infty} f(n) = \infty$.

If there exists a subsequence $\{n_i\}_{i=1}^{\infty}$ such that for all $i = 1, 2, \ldots$ and $j = 2, 3, \ldots, n$

$$b_{1,n_i} \geq b_{j,n_i}^{\tau}$$

and

$$b_{j-1,n_i} \geq r\, b_{j,n_i}$$

and

$$b_{j,n_i+1} \geq b_{1,n_i}^{g(i)},$$

then the ξ_i are algebraically independent over $\mathbb{Q}$.

AlgebraicIndependenceOfNNumbers3

Let ξ be an irrational number with regular continued fraction expansion

$$\xi = b_0 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

with unbounded partial quotients b_j. If there exist n positive integers $g_i \geq 2$, $i = 1, 2, \ldots, n$, then the n numbers x_i, $i = 1, 2, \ldots, n$.

$$x_i = (g_i - 1) \sum_{j=1}^{\infty} g_i^{-\lfloor j\,\xi \rfloor}$$

are algebraically independent over $\mathbb{Q}$.

AlgebraicIndependenceOfTwoContinuedFractions

Let ξ and η be two continued fractions

$$\xi = b_0^{(\xi)} + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j^{(\xi)}}$$

$$\eta = b_0^{(\eta)} + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j^{(\eta)}}$$

with $b_j^{(\xi)}, b_j^{(\eta)} \in \mathbb{Z}^+$.

Let $r > 1$, $\{n_j\}_{j=1}^{\infty}$ be a sequence of increasing positive integers, $f(n)$ be an integer-valued function for integer argument n, and $\lim_{j\to\infty} f(n_j) = \infty$.

Then if for all $n \in \mathbb{Z}^+$

$$\frac{b_n^{(\xi)}}{r} \geq b_n^{(\eta)} \geq \left(b_{n-1}^{(\xi)}\right)^{f(n-1)},$$

ξ and η are algebraically independent over $\mathbb{Q}$.

AlgebraicIndependenceOfTwoNumbers

Let ξ be an irrational number with regular continued fraction expansion

$$\xi = b_0 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}.$$

If there exist two positive integers $g_1 \geq g_2 > 1$ such that for all $j \geq 1$

$$b_n \geq 1 + 2 \frac{\ln(g_1)}{\ln(g_2)}$$

then the two numbers x_1 and x_2

$$x_1 = \sum_{j=1}^{\infty} (g_1 - 1)\, g_1^{\lfloor j\,\xi \rfloor}$$

$$x_2 = \sum_{j=1}^{\infty} (g_2 - 1)\, g_2^{\lfloor j\,\xi \rfloor}$$

are algebraically independent over $\mathbb{Q}$.

AlgebraicNumberContinuedFractionTermBounds

Let ξ be an algebraic number with minimal polynomial $P(x)$ of degree d with regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with A_k / B_k the sequence of its convergents. Then there exists an $m > 0$ such that for all $n > m$

$$b_n < |P'(\xi)| \, B_{m-1}^d.$$

Algorithm:AyresBackwardMethod

Given the partial denominators b_n of a regular continued fraction

$$\xi = b_0 + \underset{n=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_n}$$

the value of ξ can be computed by letting

$$P_N = b_N$$

and iterating

$$P_n = b_n \, P_{n+1} + P_{n+2}$$

from $n = N - 1$ to $n = 0$. The value of ξ is then given by

$$\xi = \frac{P_0}{P_1}.$$

Algorithm:AyresForwardMethodRationalNumber

Given a rational number $r = p / q$, the partial denominators b_n of the finite regular continued fraction

$$\xi = b_0 + \underset{n=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_n}$$

of r can be computed by setting $P_{-1} = p$, $P_0 = q$ and iterating

$$b_n = \left\lfloor \frac{P_{n-1}}{P_n} \right\rfloor$$

$$P_{n+1} = P_{n-1} - b_n \, P_n$$

starting with $n = 0$ until $P_N = 1$ and then taking $b_N = P_{N-1}$.

Algorithm:AyresForwardMethodSurd

Given an irrational square root $\sqrt{N}$, its continued fraction

$$\xi = b_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

can be calculated by iterating

$A_n = b_{n-1}\, B_{n-1} + C_{n-2}$

$B_n = b_{n-1}\, C_{n-1} + D_{n-2}$,

where

$C_n = A_n \bmod B_n$

$$b_n = \left\lfloor \frac{A_n}{B_n} \right\rfloor$$

$D_n = B_n \bmod C_n$

until $A_n = 2\, m$ and $B_n = k$ when the process repeats where $N = m^2 + k$.

Algorithm:AyresMethodLinearDiophantineEquations

Let a, c, d, x, y be integers where

$\gcd(a, c) = 1$

$a\, x = c\, y + d$.

Then one can find a solution for x and y by computing the continued fraction for a/c, finding its representation as

$$\xi = a_0 + \underset{n=1}{\overset{N}{\mathrm{K}}} \frac{1}{a_n}$$

where N is even (for odd representations can be extended with $a_N = 1$) and taking the numerator and denominator of

$$p/q = a_0 + \underset{n=1}{\overset{N}{\mathrm{K}}} \frac{1}{a_n}$$

yields a solution to

$a\, p = c\, q + 1$

and so one can set $x = d\, p$ and $y = d\, q$.

Algorithm:BackwardAlgorithm

Let ξ be the finite continued fraction of a rational number x

$$\xi = b_0 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{a_j}{b_j}.$$

The backward algorithm calculates the value of ξ through the recurrence relation

$$Q_n = \frac{a_n}{b_n}$$

$$Q_{k-1} = \frac{a_{k-1}}{b_{k-1} + Q_k}$$

for $k = n, n-1, \ldots, 1$, and the value is $\xi = b_0 + Q_1$.

Algorithm:BackwardAlgorithmRegular

Let ξ be the finite regular continued fraction of a rational number x

$$\xi = b_0 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{1}{b_j}.$$

The backward algorithm calculates the value of ξ through the recursion relation

$$Q_n = b_n$$

$$Q_{k-1} = \frac{1}{b_{k-1} + Q_k}$$

for $k = n, n-1, \ldots, 1$, and the value is $\xi = b_0 + Q_1$.

Algorithm:ChisholmContinuedFractionSolutionOfRiccatiODE

The general Riccati differential equation

$$y'(x) = a_0(x) + a_1(x)\, y(x) + a_2(x)\, y(x)^2$$

can be transfomed into reduced form

$$z'(x) = b_0(x) - z(x)^2$$

using the transformation

$$y(x) = -\frac{a_2(x)\,(a_1(x) + 2\, z(x)) + a_2{}'(x)}{2\, a_2(x)^2},$$

where

$$b_0(x) = \frac{1}{4\, a_2(x)^2}\Big(a_2{}''(x) + 2\, a_1(x)\, a_2(x)\, a_2{}'(x) + 3\, a_2{}'(x)^2 + a_1(x)^2\, a_2(x)^2 - 2\, a_2(x)\,(a_2(x)\,(a_1{}'(x) + 2\, a_0(x)\, a_2(x))\Big).$$

The solution of the reduced Riccati equation

$$z'(x) = b_0(x) - z(x)^2$$

can be expressed as a continued fraction in the form

$$z(x) = C + \underset{k=0}{\overset{\infty}{\mathrm{K}}} \frac{b_k(x) - C^2}{\frac{1}{2}\, b_k'(x)\left(b_k(x) - \alpha^2\right) + 2\, C},$$

where C is the differential equation's constant of integration. The $b_k(x)$ obey the following recursion relation:

$$b_k(x) = b_{k-1}(x) + \frac{C\, b_{k-1}'(x)}{b_{k-1}(x) - C^2} + \frac{3\, b_{k-1}'(x)^2}{4\left(b_{k-1}(x) - C^2\right)^2} - \frac{b_{k-1}''(x)^2}{2\left(b_{k-1}(x) - C^2\right)}.$$

Algorithm:CoefficientsOfStieltjesFractionForBinetsFunction

Let $J(z) = \ln(\Gamma(z)) + z - (z - 1/2)\ln(z) - 1/2\ln(2\,\pi)$ be the Binet function. Then the coefficient a_k of its S-fraction

$$J(z) = \underset{k=0}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{z}$$

obey the following recurrence relation:

$$a_0 = c_0$$

$$a_p = \frac{\sum_{j=0}^{p-1} \delta_{j/2 \bmod 2}\, c_{j/2}\left[u^j\right](q_p)}{c_0 \prod_{j=0}^{p-1} a_j}$$

where

$$q_0 = c_0$$

$$q_p = u\, q_{p-1} - a_{p-1}\, q_{p-2}$$

$$c_p = \left| \frac{B_{2p+2}}{(2p+1)(2p+2)} \right|$$

and $[u^n](q)$ denotes the coefficient of u^n in the polynomial q.

The a_k are all rational numbers and the first are:

$$a_0 = \frac{1}{12}$$

$$a_1 = \frac{1}{30}$$

$$a_2 = \frac{53}{210}$$

$$a_3 = \frac{195}{371}$$

$$a_4 = \frac{22\,999}{22\,737}$$

$$a_5 = \frac{29\,944\,523}{19\,733\,142}$$

$$a_6 = \frac{109\,535\,241\,009}{48\,264\,275\,462}$$

$$a_7 = \frac{29\,404\,527\,905\,795\,295\,658}{9\,769\,214\,287\,853\,155\,785}$$

$$a_8 = \frac{455\,377\,030\,420\,113\,432\,210\,116\,914\,702}{113\,084\,128\,923\,675\,014\,537\,885\,725\,485}$$

$$a_9 = \frac{26\,370\,812\,569\,397\,719\,001\,931\,992\,945\,645\,578\,779\,849}{5\,271\,244\,267\,917\,980\,801\,966\,553\,649\,147\,604\,697\,542}$$

$$a_{10} = \frac{152\,537\,496\,709\,054\,809\,881\,638\,897\,472\,985\,990\,866\,753\,853\,122\,697\,839}{24\,274\,291\,553\,105\,128\,438\,297\,398\,108\,902\,195\,365\,373\,879\,212\,227\,726}$$

$$a_{11} = 100\,043\,420\,063\,777\,451\,042\,472\,529\,806\,266\,909\,090\,824\,649\,341\,814\,868\,347\,109\,676\,190\,691 / 13\,346\,384\,670\,164\,266\,280\,033\,479\,022\,693\,768\,890\,138\,348\,905\,413\,621\,178\,450\,736\,182\,873$$

$$a_{12} = 76\,505\,453\,770\,729\,679\,546\,978\,925\,279\,947\,999\,751\,358\,882\,390\,333\,162\,643\,791\,755\,779\,220\,628\,608\,937\,055\,725 / 8\,462\,374\,626\,124\,882\,026\,566\,154\,328\,209\,420\,711\,352\,946\,133\,738\,527\,825\,697\,131\,889\,768\,847\,210\,043\,866\,097.$$

Algorithm:ContinuedFractionExpansionByExcess

Let x be a real number. Then the by-excess continued fraction expansion

$$\xi = b_0 + \underset{j=1}{\overset{N}{\mathrm{K}}} \frac{-1}{b_j}$$

(where N is possibly infinity) can be calculated through the repeated application of the generalized Gauss map τ: [0, 1 → [0, 1

$$\tau(x) = \left\lceil \frac{1}{x} \right\rceil - \frac{1}{x}$$

through

$$b_0 = \lceil x \rceil$$

$$b_j = \left\lceil \frac{1}{\tau^j(x)} \right\rceil.$$

Algorithm:ContinuedFractionExpansionRegular

Let x be a real number. Then the regular continued fraction expansion

$$\xi = b_0 + \underset{j=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_j}$$

(where N is possibly infinity) can be calculated through the repeated application of the Gauss map τ: [0, 1 → [0, 1

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

through

$$b_0 = \lfloor x \rfloor$$

$$b_j = \left\lfloor \frac{1}{\tau^j(x)} \right\rfloor.$$

Algorithm:EulerMindingSummationAlgorithm

Let ξ be the finite continued fraction of a rational number x

$$\xi = b_0 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{a_j}{b_j}.$$

The forward algorithm calculates the value of ξ through the recursion

$B_{-1} = 0$

$B_0 = 1$

$B_k = b_k\, B_{k-1} + a_k\, B_{k-2}$

and is given as

$$\xi = b_0 - \sum_{k=1}^{n} \frac{\prod_{j=1}^{k} (-a_j)}{B_{k-1}\, B_k}.$$

Algorithm:EulerMindingSummationAlgorithmRegular

Let ξ be the finite regular continued fraction of a rational number x

$$\xi = b_0 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{1}{b_j}.$$

The forward algorithm calculates the value of ξ through the recursion

$B_{-1} = 0$

$B_0 = 1$

$B_k = b_k\, B_{k-1} + B_{k-2}$

and is given as

$$\xi = b_0 - \sum_{k=1}^{n} \frac{(-1)^k}{B_{k-1}\, B_k}.$$

Algorithm:FareyProcess

Start with a Farey pair a/b and c/d and take their mediant M_0. Inserting M_0 into the Farey interval $I_0 = [a/b,\, c/d]$ yields two Farey subintervals $I_1^1 = [a/b,\, M_0]$ and $I_1^2 = [M_0,\, c/d]$, thus completing step one. For step two, create the mediants M_1^1 and M_1^2 of I_1^1 and I_1^2, respectively, whereby four Farey subintervals I_2^j, $j \in \{1, 2, 3, 4\}$, result. Continuing inductively, at the kth step, there will be 2^k mediants M_k^j, one for each of the 2^k Farey subintervals I_k^j, $j = 1, 2, \ldots, 2^k$.

Algorithm:FareyProcessZeroed

Given a particular number α lying in a Farey interval, a modification of the Farey process can be made in which one "zeroes in on α" by dividing said interval into Farey subintervals. This is done by inserting the mediant into the original Farey interval, whereby two subintervals are created, and considering only the resulting subinterval containing α. Then, the process is repeated inductively until approximations suitably close to α are obtained.

More precisely, let α be a number lying in some Farey interval $I_0 = [a/b, c/d]$. Form the mediant $M_0 = (a+c)/(b+d)$ and insert it into I_0, resulting in two subintervals $I_1^1 = [a/b, M_0]$ and $I_1^2 = [M_0, c/d]$. At this junction, $\alpha \in I_1^j$ for $j \in \{1, 2\}$. Assuming $\alpha \in I_1^j$, form the mediant $M_1 = (a + c_0)/(b + d_0)$ where $c_0/d_0 = M_0$ and consider the resulting Farey intervals $I_2^1 = [a/b, M_1]$ and $I_2^2 = [M_1, M_0]$. Continue inductively, whereby at the kth iteration there exist two Farey subintervals I_{k-1}^1 and I_{k-1}^2 with $\alpha \in I_k^j$, $j \in \{1, 2\}$, and M_k the mediant of I_k^j.

If $\alpha = p/q$ is rational in lowest terms, then this process terminates and α appears as an endpoint of a Farey pair at some stage of the process. If instead α is irrational, then this process can be continued ad infinitum until a rational approximation within a specified error bounds is obtained.

Algorithm:FastContinuedFractionAlgorithm

The fast continued fraction algorithm is a modified version of the zeroed Farey process in which some information calculated as part of the latter is discarded in exchange for asymptotic speed. In particular, note that for a given x (generally irrational), the zeroed Farey algorithm performs a "zeroing in" process by way of creating a series of shrinking Farey intervals containing x, each of whose endpoints are recorded as best left and right rational approximations to x. The fast continued fraction algorithm gains computational speed by recording only the last such "zeroing in" when successive shrinkings occur on one side of x or the other.

To be more precise: Start with an irrational number x in some Farey interval $[a/b, c/d]$. In the zeroed Farey process, it may happen that a succession $a_1/b_1, a_2/b_2, \ldots, a_k/b_k$ of iterations occur to zero in on x from (without loss of generality) the left; in the slow algorithm, all $2k$ of these integers would be recorded whereas in th fast algorithm, computational methods are applied to determine only the kth values a_k, b_k so as to eliminate computational overhead. As part of the fast algorithm, a "stopping index" s is computed and maintained to provide a guaranteed stopping point to the otherwise-infinite algorithm.

Here are the tools needed to implement the fast algorithm. Again, x is assumed throughout to be an irrational number lying in the Farey interval $[a/b, c/d]$.

(i) For each k, a_{k+1}/b_{k+1} is the mediant of the interval $[a_k/b_k, c/d]$. Therefore, one can compute a_k, b_k: $a_k = a + k\,c$, $b_k = b + k\,d$.

(ii) Consider the function $f(z) = (a + z\,c)/(b + z\,d)$ and note that the real number y for which $f(y) = x$ satisfies $y = (x\,b - a)/(c - x\,d)$. See the pseudo-code below.

(iii) The stopping index s is defined by $s = \lfloor y \rfloor$.

(iv) Redefine y recursively: $y = 1/(y - s)$.

The following pseudocode describes this process in more explicit detail. The variables need are a, b, c, d, y, s, a_s, and b_s. Further, x as above represents the number being approximated and n a positive integer denotes some prescribed number of iterations to perform.

```
Loop {
 s = floor (y);

 a_s = a + sc;
 b_s = b + sd;
 Print : s, a_s, b_s;

 a = c;
 b = d;
 c = a_s;
 d = b_s;
 y = 1/(y - s);
 } Until {s = n}
```

Algorithm:ForwardAlgorithm

Let ξ be the finite continued fraction of a rational number x

$$\xi = b_0 + \underset{j=1}{\overset{n}{\mathrm{K}}} \frac{a_j}{b_j}.$$

The forward algorithm calculates the value A_n / B_n of ξ through the recursion relation

$A_{-1} = 1$

$A_0 = b_0$

$A_k = b_k\, A_{k-1} + a_k\, A_{k-2}$

$B_{-1} = 0$

$B_0 = 1$

$B_k = b_k\, B_{k-1} + a_k\, B_{k-2}$,

and the value is $\xi = b_0 + A_n / B_n$.

Algorithm:ForwardAlgorithmRegular

Let ξ be the finite regular continued fraction of a rational number x

$$\xi = b_0 + \underset{j=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_j}.$$

The forward algorithm calculates the value A_n / B_n of ξ through the recursion relation

$A_{-1} = 1$

$A_0 = b_0$

$A_k = b_k\, A_{k-1} + A_{k-2}$

$B_{-1} = 0$

$B_0 = 1$

$B_k = b_k\, B_{k-1} + B_{k-2}$,

and the value is $\xi = b_0 + A_n / B_n$.

Algorithm:GosperRegularContinuedFractionArithmetic

Given two regular continued fraction expansions for real numbers A and B

$$A = a_0 + \underset{k=1}{\overset{n_A}{\mathrm{K}}} \frac{1}{a_k}$$

$$B = b_0 + \overset{n_B}{\mathrm{K}} \frac{1}{}$$

$\underset{k=1}{} \; b_k$

(with n_A and/or n_B possibly ∞), an arithmetic operation f (addition, subtraction, multiplication, and division) can be carried out on the sequences of partial denominators $\{a_k\}_{k=0}^{n_A}$ and $\{b_k\}_{k=0}^{n_B}$ directly to obtain the partial denominators c_k of

$$C = f(A, B) = c_0 + \underset{k=1}{\overset{n_C}{\mathrm{K}}} \frac{1}{c_k}.$$

More generally, the partial denominators c_k of the expression

$$C = f(A, B) = \frac{a\,A\,B + b\,A + c\,B + d}{e\,A\,B + f\,A + g\,B + h}$$

(with the special cases for two continued fractions

addition	$a = 0$	$b = 1$	$c = 1$	$d = 0$	$e = 0$	$f = 0$	$g = 0$	$h = 1$
subtraction	$a = 0$	$b = 1$	$c = -1$	$d = 0$	$e = 0$	$f = 0$	$g = 0$	$h = 1$
multiplication	$a = 1$	$b = 0$	$c = 0$	$d = 0$	$e = 0$	$f = 0$	$g = 0$	$h = 1$
division	$a = 0$	$b = 1$	$c = 0$	$d = 0$	$e = 0$	$f = 0$	$g = 1$	$h = 1$

for the basic arithmetic operations)

for given rational expressions a, b, c, d, e, f, g can be computed directly from the partial denominator sequences $\{a_k\}_{k=0}^{n_A}$ and $\{b_k\}_{k=0}^{n_B}$.

Observing that the expression

$$\frac{a\,A\,B + b\,A + c\,B + d}{e\,A\,B + f\,A + g\,B + h}$$

(i) under the substitution $A \to a_k + 1/A$ changes as

$$\frac{a\left(a_k + \frac{1}{A}\right) B + b\left(a_k + \frac{1}{A}\right) + c\,B + d}{e\left(a_k + \frac{1}{A}\right) B + f\left(a_k + \frac{1}{A}\right) + g\,B + h} = \frac{(c + a\,a_k)\,A\,B + (d + b\,a_k)\,A + a\,B + b}{(g + e\,a_k)\,A\,B + (h + f\,a_k)\,A + e\,B + f},$$

(ii) under the substitution $B \to b_k + 1/B$

$$\frac{a\,A\left(b_k + \frac{1}{B}\right) + b\,A + c\left(b_k + \frac{1}{B}\right) + d}{e\,A\left(b_k + \frac{1}{B}\right) + f\,A + g\left(b_k + \frac{1}{B}\right) + h} = \frac{(b + a\,b_k)\,A\,B + a\,A + (d + c\,b_k)\,B + c}{(f + e\,b_k)\,A\,B + A\,e + (h + g\,b_k)\,B + g},$$

(iii) and

$$\frac{1}{\frac{a\,A\,B + b\,A + c\,B + d}{e\,A\,B + f\,A + g\,B + h} - c_k} = \frac{e\,A\,B + f\,A + g\,B + h}{(a - e\,c_k)\,A\,B + (b - f\,c_k)\,A + (c - g\,c_k)\,B + (d - h\,c_k)}$$

shows the shape invariance of the expression

$$\frac{a\,A\,B + b\,A + c\,B + d}{e\,A\,B + f\,A + g\,B + h}$$

under the operations (i), (ii), and (iii).

The two substitutions $A \to a_k + 1/A$ and $B \to b_k + 1/B$ can be thought as using the kth partial denominators and denoting the remainders by the symbolic variable A or B.

The operation (iii) can be interpreted as extracting the kth digit c_k from $F(A, B)$. Observing that substituting $A \to a_k + 1/A$ and $B \to b_k + 1/B$ repeatedly (a_k, and b_k are positive integers from the regular continued fraction expansions of A and B) into an expression of the form

$$\frac{a\,A\,B + b\,A + c\,B + d}{e\,A\,B + f\,A + g\,B + h}$$

and denoting the result of this substitution by

$$g\left(\{a, b, c, d, e, f, g\}, \{a_k, a_{k+1}, \ldots, a_{k+m}\}, \left\{b_j, b_{j+1}, \ldots, b_{j+n}\right\}\right)$$

and taking into account that the remainders A and B are bounded from below by 1, allows to bound this expression from above and below. For sufficiently large m and n, there exists an integer ω such that

$$\omega \leq g\left(\{a, b, c, d, e, f, g\}, \{a_k, a_{k+1}, \ldots, a_{k+m}\}, \left\{b_j, b_{j+1}, \ldots, b_{j+n}\right\}\right) < \omega + 1.$$

Then ω is the next partial denominator of the regular continued fraction expansion of $f(A, B)$.

So, applying (possibly multiple times) (i) and (ii) and then (iii) repeatedly, allows to extract a continued fraction digit from $F(A, B)$. This process can be repeated to obtain the sequence of partial denominators $\{c_k\}_{k=0}^{n_C}$.

Algorithm:HurwitzExpansion

Let z be a complex number. Then the Hurwitz continued fraction expansion

$$z = b_0 + \underset{j=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_j}$$

(where N is possibly infinity) can be calculated through the repeated application of the map

$$\tau(\zeta) = \frac{1}{\zeta} - \left\lfloor \frac{1}{\zeta} \right.$$

through

$$b_0 = \lfloor z$$

$$b_j = \left\lfloor \frac{1}{\tau^j(z)} \right..$$

Here, $\lfloor z$ denotes rounding to the nearest Gaussian integer.

Algorithm:JacobiPerronAlgorithm

Given a list of d ($d > 1$) real numbers $\{\alpha_1, \alpha_2, \ldots, \alpha_d\}$, the Jacobi-Perron algorithm calculates a multidimensional continued fraction that simultaneously approximates the given real numbers.

Start setting:

$\alpha_i^{(0)} = \alpha_i$ for $1 \leq i \leq d$.

Define

$a_i^{(n)} = \lfloor \alpha_i^{(n)} \rfloor$ for $1 \leq i \leq d - 1$ and $n \geq 1$.

Recursively define

$$\alpha_d^{(n)} = \frac{1}{\alpha_1^{(n-1)} - a_1^{(n-1)}}$$

$\alpha_i^{(n)} = \alpha_d^{(n)} \left(\alpha_{i+1}^{(n-1)} - a_{i+1}^{(n-1)}\right)$ for $2 \leq i \leq d$.

Then the simultaneous approximations

$$\alpha_i \approx \frac{p_i^{(n)}}{q^{(n)}} \text{ for } 1 \leq i \leq d$$

can be obtained from

$\mathbb{A}_n = \mathbb{1}_{d+1} \cdot \mathbb{B}_1 \cdot \mathbb{B}_2 \cdot \ldots \cdot \mathbb{B}_{n-1}$

where

$$\mathbb{B}_n = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & a_1^{(n)} \\ 0 & 1 & \ldots & 0 & a_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & a_d^{(n)} \end{pmatrix}$$

and

$$\mathbb{A}_n = \begin{pmatrix} q^{(n-d)} & q^{(n-d+1)} & \ldots & q^{(n-1)} & q^{(n)} \\ p_1^{(n-d)} & p_1^{(n-d+1)} & \ldots & p_1^{(n-1)} & p_1^{(n)} \\ p_2^{(n-d)} & p_2^{(n-d+1)} & \ldots & p_2^{(n-1)} & p_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_d^{(n-d)} & p_d^{(n-d+1)} & \ldots & p_d^{(n-1)} & p_d^{(n)} \end{pmatrix}.$$

If $\alpha_i^{(n)} \in \mathbb{Z}^+$ for some n and i the algorithm is interrupted and continued with the remaining $\alpha_i^{(n)}$.

Algorithm:LangTrotterAlgorithm

The Lang-Trotter algorithm is a method of finding the continued fraction expansion of irrational roots of certain classes of polynomials by way of constructing a series of related polynomials, each having a few very specific properties, the roots of which yield the partial quotients for the aforementioned continued fraction. Among other benefits, the Lang-Trotter algorithm has the boasts the ability to find the partial quotients to with full precision and no rounding errors due to its utilization of strictly integer arithmetic.

To begin, start with a polynomial $p_n(x)$ of degree d which has positive leading coefficient and a single, simple irrational root $y_n > 1$. The process to construct the first related polynomial $p_{n+1}(x)$ is as follows. Let $a_n = \lfloor y_n \rfloor$ denote the integer part of y_n and note that by definition, a_n is the greatest integer for which $p_n(a_n) < 0$. From this, define the polynomials $Q_n(x) = P_n(x + a_n)$ and $P_{n+1}(x) = -x^d Q_n\left(x^{-1}\right)$. Because $a_n = \lfloor y_n \rfloor$, it follows that Q_n has a single root at the value $y_n - a_n \in (0, 1)$. Moreover, because the root of P_{n+1} is the reciprocal of the root of Q_n, P_{n+1} again has a single root y_{n+1} which itself is simple, irrational, and greater than 1 and which has the form $y_{n+1} = (y_n - a_n)^{-1}$. Also note that because the constant term of Q_n is negative, the leading coefficient of P_{n+1} will again be positive, whereby it follows that P_{n+1} has all the properties assumed for P_n.

Therefore, the above process can be repeated, and so beginning with a polynomial P_1 with the properties assumed initially, an infinite sequence $P_1(x), P_2(x), \ldots$ of polynomials can be formed which all have those assumed properties and which have roots $y_1, y_2, \ldots$. Subsequently, the sequence $a_1, a_2, \ldots$ is the sequence of partial quotients in the continued fraction expansion ξ_1 of y_1 where $a_n = \lfloor y_n \rfloor$. Moreover, because the above process consists only of integer addition and multiplication, it follows that no rounding errors, etc., are introduced throughout so that full precision results are obtained.

Algorithm:ModifiedLentAlgorithm

Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n}$$

be a generalized continued fraction, a_n be the partial numerator of ξ, b_n be the partial denominator of ξ, c_n be a sequence, d_n be a sequence, and f_n be a sequence. Given

$$c_0 = b_0$$

$$d_0 = 0$$

$$c_{1+n} = b_{1+n} + \frac{a_{1+n}}{c_n}$$

$$d_{1+n} = \frac{1}{b_{1+n} + \frac{a_{1+n}}{d_n}}$$

$$f_{1+n} = c_{1+n}\, d_{1+n}\, f_n$$

then

$$\xi = \lim_{n\to\infty} f_n.$$

Algorithm:NearestIntegerContinuedFractionExpansion

Let ξ be a real number. Then the nearest integer continued fraction expansion

$$x = \varepsilon_0\, b_0 + \underset{j=1}{\overset{N}{\mathrm{K}}} \frac{\varepsilon_j}{b_j}$$

(where N is possibly infinity), $\varepsilon_j \in \{-1, 1\}$, and $b_j \in \mathbb{Z}^+$ can be calculated through the repeated application of the map τ: [$-1/2$, $1/2 \to$ [$-1/2$, $1/2$

$$\tau(x) = \frac{\operatorname{sgn}(x)}{x} - \left\lfloor \frac{\operatorname{sgn}(x)}{x} + \frac{1}{2} \right\rfloor$$

$$\tau(0) = 0$$

through

$$b_0 = \left\| \left\lfloor x + \frac{1}{2} \right\rfloor \right\|$$

$$\varepsilon_0 = \operatorname{sgn}\!\left(\left\lfloor x + \frac{1}{2} \right\rfloor\right)$$

$$\varepsilon_j = \operatorname{sgn}(\tau^n(x))$$

$$b_j = \left\lfloor \frac{\operatorname{sgn}(\tau^n(x))}{\tau^n(x)} + \frac{1}{2} \right\rfloor.$$

Here $b_j \ge 2$ for $n \ge 1$ and $b_j + \varepsilon_{j+1} \ge 2$ for $n \ge 1$.

Algorithm:OstrowskiNumberSystemIntegers

Let ξ be the positive irrational number $0 < \xi < 1$ with regular continued fraction expansion

$$\xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

and convergents A_n / B_n.

For every irrational number ξ with $0 < \xi < 1$, any integer n can be uniquely written as

$$N = \sum_{k=1}^{m} c_k B_{k-1}$$

where

$0 \leq c_1 \leq b_1 - 1$

$0 \leq c_1 \leq b_1$ for $k \geq 2$

$c_k = 0$ if $c_{k+1} = b_{k+1}$.

The Ostrowski digits c_k can be obtained recursivley in the following manner:

1) Determine m such that $B_{m+1} > N$.

2) Define the c_k recursively starting with

$$c_m = \left\lfloor \frac{N}{B_m} \right\rfloor$$

$$\delta_m = N - c_m B_m$$

and for $k < m$ through

$$c_k = \left\lfloor \frac{\delta_{k+1}}{B_m} \right\rfloor$$

$$\delta_k = \delta_{k+1} - c_k B_k.$$

Algorithm:PippengerContinuedFraction

Any real number $1 \le \xi \le 2$ can be expressed as a Pippenger continued fraction

$$\xi = 1 + \frac{1}{-1 + t_1\left(1 + \frac{1}{-1+t_2\left(1+\frac{1}{-1+t_3(\ldots)}\right)}\right)}$$

where $t_k \in \mathbb{Z}^+$ and $t_k \ge 2$. Then t_k can be calculated recursively as long as $y_k > 1$ through

$$y_0 = \xi$$

$$y_{k+1} = \frac{z_{k+1}}{t_{k+1}}$$

$$z_{k+1} = 1 + \frac{1}{y_k - 1}$$

$$t_k = \lfloor z_k \rfloor.$$

Algorithm:ProgressiveRutishauserQD

The reciprocal of the formal power series

$$f(z) = \sum_{k=0}^{\infty} d_k z^k$$

with $c_k \in \mathbb{C}$ can be converted into a regular C-fraction

$$\frac{1}{f(z)} = d_0 - \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k z}{1}$$

with $a_k \in \mathbb{C} \setminus 0$ for $k \ge 1$.

Assuming the C-fraction exists, the a_k are given by

$$a_k = \begin{cases} d_1 & \text{for } k = 1 \\ -q_{k/2}^{(1)} & \text{for } k/2 \in \mathbb{Z} \\ -e_{(k-1)/2}^{(1)} & \text{for } (k-1)/2 \in \mathbb{Z}. \end{cases}$$

The coefficients $q_k^{(l)}$ and $e_k^{(l)}$ can be recursively calculated through

$$e_0^{(-1)} = 0$$

$$e_1^{(0)} = \frac{d_2}{d_1}$$

$$q_1^{(0)} = -\frac{d_1}{d_0}$$

$$e_l^{(k)} = \frac{q_{l+1}^{(k-1)}}{q_l^{(k)}} e_l^{(k-1)} \text{ for } k \ge -l$$

$$q_l^{(k)} = q_l^{(k-1)} + e_l^{(k-1)} - e_{l-1}^{(k)} \text{ for } k \ge -l\,.$$

Algorithm:RosenShallitAlgorithm

The Rosen-Shallit algorithm is a procedure for identifying and computing the complete list of roots of a polynomial with integer coefficients. The algorithm itself is itself a composition of other algorithms and theorems including Uspensky's algorithm, Newton's method, Vincent's theorem, and others. The breakdown of the procedure is as follows.

To begin, start with a polynomial $p(x)$ with real coefficients and let $\epsilon > 0$ be an error tolerance for the approximations of the irrational roots of p. The steps for the algorithm are:

1. Test $p(x)$ for rational roots and their multiplicities using the rational root theorem. Factor them out and consider the remaining polynomial $\hat{p}(x)$ whose real roots are all irrational.

2. Use Uspensky's algorithm to test $\hat{p}(x)$ for multiple roots and use the algorithm to factor $\hat{p}$ so that $\hat{p}\,(x) = a_0\, X_1\, X_2^2 \cdots X_r^r$ where $a_0 \in \mathbb{R}$ is a constant and where, for $i = 1,\, 2,\, \ldots\,,\, r$,

$X_i = (x - b_1)\,(x - b_2) \cdots \left(x - b_j\right)$

is a polynomial whose simple roots $b_1,\, b_2,\, \ldots\,,\, b_j$ are all the roots of multiplicity of i of $\hat{p}(x)$.

3. Use Vincent's theorem to separate the roots $b_{k,1},\, b_{k,2},\, \ldots\,,\, b_{k,j}$ of each factor X_k of $\hat{p}$, $k = 1,\, 2,\, \ldots\,,\, r$. Using the transformation defined in the theorem, find for each $b_{k,j}$ a polynomial $\hat{p}_{k,j}(x)$ having $b_{k,j}$ as its only positive root.

4. For each $\hat{p}_{k,j}(x)$, use Newton's method to find an initial approximation for the root $b_{k,j}$. Given this initial approximation, use the Lang-Trotter algorithm to compute the partial quotients of the approximants A_n / B_n of the continued fraction representation $\xi_{k,j}$ of $b_{k,j}$.

5. Conclude the process at the nth approximant A_n / B_n whenever $1 \big/ B_n^2 < \epsilon$.

6. Find any negative roots of $p(x)$ by performing the above process on the polynomial $p(-x)$.

The authors make note of the fact that very little is known about the computational efficiency of their algorithm, noting only that smaller values for ϵ yields slowing of computation; they also note that the accuracy of their output agrees with that of Vincent on comparable polynomials. Theoretically, the inclusion of the Lang-Trotter algorithm, which itself is computationally more efficient than other, more brute-force methods, improves both computational efficiency and accuracy due to the lack of roundoff error involved. Moreover, the inclusion of Newton's method reduces the number of computations needed for step 4 by requiring each iteration to test only three integers for each polynomial's sign change versus testing $y_m + 1$ integers for the brute force alternative described in their paper. Here, y_m is the root of the polynomial P_m formed in the mth step of the Lang-Trotter algorithm.

Algorithm:SchmidtExpansion

Let ξ be a complex number with $\mathrm{Im}(\xi) \geq 0$. The Schmidt continued fraction expansion

$\xi = M_1 \cdot M_2 \cdot \ldots \cdot M_N$

(where N is possibly infinity) with complex 2×2 matrices

$M_k \in \{V_1, V_2, V_3, C, E_1, E_2, E_3\}$ where

$$V_1 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 1-i & i \\ -i & 1+i \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1+i \\ 1-i & i \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & -1+i \\ 0 & i \end{pmatrix}$$

$$E_3 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

can be calculated through the repeated application of the map

$$\tau : \{\{z : z \in \mathbb{C} \bigwedge \mathrm{Im}(z) \geq 0\}, \{0, 1\}, \{V_1, V_2, V_3, C, E_1, E_2, E_3\}\} \to \{\{z : z \in \mathbb{C} \bigwedge \mathrm{Im}(z) \geq 0\}, \{0, 1\}, \{V_1, V_2, V_3, C, E_1, E_2, E_3\}\}$$

$$\tau(z, \varepsilon, M) = \begin{cases} \{m(z, V_1^{-1}), \varepsilon, V_1\} & (\varepsilon = 1 \bigwedge z \in \mathcal{R}(V_1)) \bigvee (\varepsilon = 0 \bigwedge z \in \mathcal{R}(V_1^*)) \\ \{m(z, V_2^{-1}), \varepsilon, V_2\} & (\varepsilon = 1 \bigwedge z \in \mathcal{R}(V_2)) \bigvee (\varepsilon = 0 \bigwedge z \in \mathcal{R}(V_2^*)) \\ \{m(z, V_3^{-1}), \varepsilon, V_3\} & (\varepsilon = 1 \bigwedge z \in \mathcal{R}(V_3)) \bigvee (\varepsilon = 0 \bigwedge z \in \mathcal{R}(V_3^*)) \\ \{m(z, E_1^{-1}), 1-\varepsilon, E_1\} & \varepsilon = 1 \bigwedge z \in \mathcal{R}(E_1) \\ \{m(z, E_2^{-1}), 1-\varepsilon, E_2\} & \varepsilon = 1 \bigwedge z \in \mathcal{R}(E_2) \\ \{m(z, E_3^{-1}), 1-\varepsilon, E_3\} & \varepsilon = 1 \bigwedge z \in \mathcal{R}(E_3) \\ \{m(z, C^{-1}), z, 1-\varepsilon, C & (\varepsilon = 1 \bigwedge z \in \mathcal{R}(C)) \bigvee (\varepsilon = 0 \bigwedge z \in \mathcal{R}(C^*)) \end{cases}$$

where the regions $\mathcal{R}$ are defined as

$$\mathcal{R}(V_1) = \{z : z \in \mathbb{C} \bigwedge \mathrm{Im}(z) \geq 1\}$$

$$\mathcal{R}(V_2) = \left\{z : z \in \mathbb{C} \bigwedge \left|z - \frac{i}{2}\right| \leq \frac{1}{2}\right\}$$

$$\mathcal{R}(V_3) = \left\{z : z \in \mathbb{C} \bigwedge \left|z - \left(1 + \frac{i}{2}\right)\right| \leq \frac{1}{2}\right\}$$

$$\mathcal{R}(C) = \left\{z : z \in \mathbb{C} \bigwedge 0 < \mathrm{Re}(z) < 1 \bigwedge\right.$$

$$\frac{1}{2} < \mathrm{Im}(z) < 1 \bigwedge \left|z - \frac{i}{2}\right| > \frac{1}{2} \bigwedge \left|z - \left(1 + \frac{i}{2}\right)\right| > \frac{1}{2}\Big\}$$

$$\mathcal{R}(E_1) = \Big\{z : z \in \mathbb{C} \bigwedge 0 < \mathrm{Re}(z) < 1 \bigwedge 0 \le \mathrm{Im}(z) < \frac{1}{2} \bigwedge$$

$$\left|z - \frac{i}{2}\right| > \frac{1}{2} \bigwedge \left|z - \left(1 + \frac{i}{2}\right)\right| > \frac{1}{2}\Big\}$$

$$\mathcal{R}(E_2) = \left\{z : z \in \mathbb{C} \bigwedge 0 < \mathrm{Re}(z) > 1 \bigwedge 0 \le \mathrm{Im}(z) < 1 \bigwedge \left|z - \left(1 + \frac{i}{2}\right)\right| > \frac{1}{2}\right\}$$

$$\mathcal{R}(E_3) = \left\{z : z \in \mathbb{C} \bigwedge \mathrm{Re}(z) < 0 \bigwedge 0 \le \mathrm{Im}(z) < 1 \bigwedge \left|z - \frac{i}{2}\right| > \frac{1}{2}\right\}$$

$$\mathcal{R}(V_1^*) = \left\{z : z \in \mathbb{C} \bigwedge 0 \le \mathrm{Re}(z) \le 1 \bigwedge \mathrm{Im}(z) > 1 \bigwedge \left|z - \left(\frac{1}{2} + i\right)\right| > \frac{1}{2}\right\}$$

$$\mathcal{R}(V_2^*) = \Big\{z : z \in \mathbb{C} \bigwedge 0 \le \mathrm{Re}(z) < \frac{1}{2} \bigwedge$$

$$0 \le \mathrm{Im}(z) \le 1 \bigwedge \left|z - \frac{1}{2}\right| \ge \frac{1}{2} \bigwedge \left|z - \left(\frac{1}{2} + i\right)\right| > \frac{1}{2}\Big\}$$

$$\mathcal{R}(V_3^*) = \Big\{z : z \in \mathbb{C} \bigwedge \frac{1}{2} < \mathrm{Re}(z) \le 1 \bigwedge 0 \le \mathrm{Im}(z) \le 1 \bigwedge$$

$$\left|z - \frac{1}{2}\right| \ge \frac{1}{2} \bigwedge \left|z - \left(\frac{1}{2} + i\right)\right| > \frac{1}{2}\Big\}$$

$$\mathcal{R}(C^*) = \left\{z : z \in \mathbb{C} \bigwedge \left|z - \left(\frac{1}{2} + i\right)\right| \le \frac{1}{2}\right\}$$

and

$$m\left(z, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \frac{a_{11}\, z + a_{12}}{a_{21}\, z + a_{22}}.$$

Let

$$P_1(z, \varepsilon, M) = z$$
$$P_2(z, \varepsilon, M) = \varepsilon,$$
$$P_3(z, \varepsilon, M) = M,$$

then the M_j in the expansion of ξ are given as

$$M_j = P_3\left(\tau^j\left(\xi, 1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right).$$

Let

$$\xi_n = M_1 \cdot M_2 \cdot \ldots \cdot M_n$$

be the truncated expansions ($n \le N$) and let

$$\begin{pmatrix} A_n^{(0)} & A_n^{(\infty)} & A_n^{(1)} \\ B_n^{(0)} & B_n^{(\infty)} & B_n^{(1)} \end{pmatrix} = \xi_n \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

then the n^{th} convergent of ξ is the element of $\left\{A_n^{(0)} \big/ B_n^{(0)},\ A_n^{(1)} \big/ B_n^{(1)},\ A_n^{(\infty)} \big/ B_n^{(\infty)}\right\}$ that is nearest to ξ.

Algorithm:StandardRutishauserQD

The formal power series

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

with $c_k \in \mathbb{C}$ can be converted into a regular C-fraction

$$f(z) = c_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k z}{1}$$

with $a_k \in \mathbb{C} \setminus 0$ for $k \geq 1$.

Assuming the C-fraction exists, the a_k are given by

$$a_k = \begin{cases} c_1 & \text{for } k = 1 \\ -q_{k/2}^{(1)} & \text{for } k/2 \in \mathbb{Z} \\ -e_{(k-1)/2}^{(1)} & \text{for } (k-1)/2 \in \mathbb{Z}. \end{cases}$$

The coefficients $q_k^{(l)}$ and $e_k^{(l)}$ can be recursively calculated through

$e_0^{(k)} = 0$ for $k \geq 1$

$q_1^{(k)} = \dfrac{c_{k+1}}{c_k}$ for $k \geq 0$

$e_l^{(k)} = q_l^{(k+1)} - q_l^{(k)} + e_{l-1}^{(k+1)}$ for $k \geq 1$ and $l \geq 1$

$q_l^{(k)} = \dfrac{e_{l-1}^{(k+1)}}{e_l^{(k+1)}} q_{l-1}^{(k+1)}$ for $k \geq 1$ and $l \geq 2$.

Algorithm:TennerAlgorithm

Let d be a squarefree integer, $x = \sqrt{d}$ be a quadratic irrational,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction of x, and a_n, P_n, Q_n, R_n be integers. Given

$P_0 = 0,$

$Q_{-1} = d,$

$Q_0 = 1,$

$R_0 = 0,$

$a_0 = \lfloor x \rfloor,$

$P_{1+n} = \lfloor x \rfloor - R_n,$

$Q_{1+n} = -(a_n (-P_n + P_{1+n})) + q(-1 + n),$

$R_{1+n} = \lfloor x \rfloor + P_{1+n} - a_{1+n} Q_{1+n}$

and

$$a_{1+n} = \left\lfloor \frac{-x + P_{1+n}}{Q_{1+n}} \right\rfloor,$$

it follows that

$a_n = b_n.$

Algorithm:ThieleContinuedFractionAlgorithm

The Thiele continued fraction algorithm for a function $f(x)$ given $n + 1$ distinct points x_j, $j = 0, 1, 2, \ldots, n$ is

$$R_n(x) = f(x_0) + \underset{j=1}{\overset{n}{\mathrm{K}}} \frac{x - x_j}{b_j}$$

where the b_j are recursively defined through

$b_j = \Phi[x_0, x_1, \ldots, x_j]$

$\Phi[x_j] = f(x_j)$

$$\Phi[x_0, x_1, \ldots, x_{j-1}, x_j, x_{j+1}] = \frac{x_{j+1} - x_j}{\Phi[x_0, x_1, \ldots, x_{j-1}, x_{j+1}] - \Phi[x_0, x_1, \ldots, x_{j-1}, x_j]}.$$

Algorithm:UspenskyAlgorithm

Given a polynomial $p(x)$, Uspensky's algorithm is a procedure by which $p(x)$ can be decomposed into the product of polynomials $X_1, X_2^2, \ldots, X_r^r$ so that for $k = 1, 2, \ldots, r$, X_k is the product of linear factors of $p(x)$ corresponding to roots of multiplicity k. More precisely, the result of performing Uspensky's algorithm on a general polynomial $p(x)$ with r multi-roots is a decomposition

$$p(x) = a_0\, X_1\, X_2^2 \cdots X_r^r$$

of $p(x)$ where $a_0 \in \mathbb{R}$ is a constant and where for $i = 1, 2, \ldots, r$,

$$X_i = (x - b_1)(x - b_2) \cdots \left(x - b_j\right)$$

is a polynomial whose simple roots $b_1, b_2, \ldots, b_j$ are all the roots of multiplicity of i of $p(x)$. For example, given

$$p(x) = (x-1)(x-2)(x-3)^2 (x-4)^2 (x-5)^3,$$

it follows that $p(x) = a_0\, X_1\, X_2^2\, X_3^3$ where $a_0 = 1$, $X_1 = (x-1)(x-2)$, $X_2 = (x-3)(x-4)$, and $X_3 = x - 5$. The process to compute this for general p is given below.

To begin, recall that p is an arbitrary polynomial with r multi-roots and define $D_1 = \gcd(P, P')$ where P' is the standard derivative of P. Similarly, let $D_2 = \gcd(D_1, D_1')$, $D_3 = \gcd(D_2, D_2')$, and for general k, $2 \le k \le r$, $D_k = \gcd\left(D_{k-1}, D_{k-1}'\right)$. Under this identification, each D_k can be expressed in terms of X_j, $1 \le k \le r$, $1 < j \le r$: In particular, $D_1 = X_2\, X_3^2\, X_4^3 \cdots X_r^{r-1}$, $D_2 = X_3\, X_4^2 \cdots X_r^{r-2}$, and for general k, $1 \le k \le r-1$,

$$D_k = X_{k+1}\, X_{k+2}^2 \cdots X_r^{r-k}.$$

It is easy to see that this identification ends with D_{r-1}, which is necessarily constant; this confirms that p has no roots whose multiplicity is greater than r.

Uspensky's algorithm will be complete if the above information can be manipulated to find explicit expressions for X_k, $k = 1, 2, \ldots, r$. To that end, consider defining a sequence $P_1, \ldots, P_r$ of polynomials by way of the following recursive formula: $P_1 = P/D_1 = X_1\, X_2 \cdots X_r$, $P_2 = D_1/D_2 = X_2\, X_3 \cdots X_r$, and for general k, $1 \le k \le r$, $P_k = D_{k-1}/D_k = X_k\, X_{k+1} \cdots X_r$. In particular, this implies $P_r = D_{r-1}/D_r = X_r$. Having created the sequence $P_1, P_2, \ldots, P_r$, explicit expressions for X_k, $k = 1, 2, \ldots, r$, can be isolated: $X_1 = P_1/P_2$, $X_2 = P_2/P_3$, and for general k, $1 \le k \le r$, $X_k = P_k/P_{k+1}$. Using the above definitions, it is easily confirmed each root of each X_k, $1 \le k \le r$, has multiplicity k, whereby the factorization (and hence the algorithm) is complete.

Algorithm:ViskovatovMethod

The expression

$$f = \frac{\sum_{k=0}^{n} f_{1,k}\, x^k}{\sum_{k=0}^{n} f_{0,k}\, x^k}$$

has the equivalent continued fraction (C-fraction) expansion

$$f \sim \frac{f_{1,0}}{f_{0,0} + \operatorname*{K}_{k=1}^{\infty} \frac{d_{k,0}\, x}{d_{k-1,0}}},$$

where

$$d_{k,i} = d_{k-2,j+1}\, d_{k-1,0} - d_{k-1,j+1}\, d_{k-2,0}$$

$$d_{0,k} = f_{0,k}$$

$$d_{1,k} = f_{1,k}$$

assuming that no relevant coefficients vanish.

The algorithm is based on the recursive application of the identity

$$\frac{\sum_{k=0}^{\infty} a_k\, x^k}{\sum_{k=0}^{\infty} b_k\, x^k} = \frac{a_0}{b_0} + \frac{x}{\frac{\sum_{k=0}^{\infty} b_k\, x^k}{\sum_{k=0}^{\infty} \left(a_{k+1} - \frac{a_0}{b_0}\, b_{k+1}\right) x^k}} = \frac{a_0}{b_0} + \frac{x}{\frac{\sum_{k=0}^{\infty} \tilde{a}_k\, x^k}{\sum_{k=0}^{\infty} \tilde{b}_k\, x^k}}.$$

AlmostEverywhereIntegralFormOfExtendedGaussMapValues

Let τ be the natural extension of the Gauss map

$$\overline{\tau} : (0,\, 1) \times [0,\, 1] \to \mathbb{R}^2$$

$$\overline{\tau}(x,\, \theta) = \left\{\tau(x),\, \frac{1}{b_1(x) + \theta}\right\}$$

where τ(x) is the Gauss map

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

and $b_1(x) = \tau(\tau(x))$.

Then for any measureable function f from $[0,\, 1] \times [0,\, 1] \to \mathbb{R}^2$ the following identity holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\overline{\tau}^n) = \frac{1}{\ln(2)} \int_0^1 \int_0^1 \frac{f(x,\, y)}{(1 + x\, y)^2}\, dx\, dy.$$

ApproximantDifferenceForRegularContinuedFractionsWithConstantPartialQuotients

Given a regular continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n, for all $n > 1$ and $n - 1 \geq r \geq 2$,

$$\frac{A_n}{B_n} - \frac{A_{n-r}}{B_{n-r}} = \frac{(-1)^{1+n+r} \sum_{i=0}^{\lfloor (r-1)/2 \rfloor} \binom{r-1-i}{i} a^{r-1-2i}}{B_n B_{n-r}}.$$

ApproximantsToIrrationalsViaFastContinuedFractionAlgorithm

Let $\alpha > 0$ be an irrational number and let $s_0, s_1, s_2, \ldots$ be the output values from the fast continued fraction algorithm with respect to α. Then α can be expressed as a continued fraction ξ of the form

$$\xi = s_0 + \frac{1}{s_1 + \frac{1}{s_2 + \frac{1}{\ddots + \frac{1}{s_{n-1} + \frac{1}{\gamma_n}}}}},$$

where $\gamma_n > 1$ is an irrational selected to make the equality hold. What is more, if γ_n is replaced by s_n, the fraction chain becomes a rational number p_n / q_n and for each $n = 1, 2, \ldots$, these p_n / q_n are the terms in the fast continued fraction algorithm for α; for n even, $p_n / q_n = a / b$ is a left approximation and if n is odd, $p_n / q_n = c / d$ is a right approximation.

ApproximateHausdorffDimensionForContinuedFractionsWithPartialDenominatorsBoundedType

Let A be a set of natural numbers, $C(A)$ be regular continued fractions whose partial denominators are in A, H be the Hausdorff dimension, and R_n be natural numbers less than or equal to n. Then $H(C(\{1, 4, 7\})) \approx \frac{2589}{50\,000}$, $H(C(\{1, 2\})) \approx \frac{166}{3125}$, $H(C(\{1, 3, 8\})) \approx \frac{2719}{50\,000}$, $H(C(\{1, 3, 7\})) \approx \frac{1383}{25\,000}$, $H(C(\{1, 3, 6\})) \approx \frac{1413}{25\,000}$, $H(C(\{1, 3, 5\})) \approx \frac{5813}{100\,000}$, $H(C(\{1, 2, 10\})) \approx \frac{5951}{100\,000}$, $H(C(\{1, 3, 4\})) \approx \frac{3021}{50\,000}$, $H(C(\{1, 2, 7\})) \approx \frac{6179}{100\,000}$, $H(C(\{1, 2, 7, 40\})) \approx \frac{1253}{20\,000}$, $H(C(\{1, 2, 5\})) \approx \frac{323}{5000}$, $H(C(\{1, 2, 5, 40\})) \approx \frac{1633}{25\,000}$, $H(C(\{1, 2, 4\})) \approx \frac{1673}{25\,000}$, $H(C(\{1, 2, 4, 40\})) \approx \frac{3377}{50\,000}$, $H(C(\{1, 2, 4, 15\})) \approx \frac{6899}{100\,000}$, $H(C(\{1, 2, 3\})) \approx \frac{441}{6250}$, $H(C(\{1, 2, 4, 7\})) \approx \frac{1437}{20\,000}$, $H(C(\{1, 2, 4, 6\})) \approx \frac{291}{4000}$, $H(C(\{1, 2, 4, 5\})) \approx \frac{37}{500}$, $H(C(\{1, 2, 3, 6\})) \approx \frac{1897}{25\,000}$, $H(C(\{1, 2, 3, 5\})) \approx \frac{7709}{100\,000}$, $H(C(\{1, 2, 3, 4\})) \approx \frac{7889}{100\,000}$, $H(C(\{1, 2, 3, 4, 10\})) \approx \frac{8081}{100\,000}$, $H(C(\{1, 2, 3, 4, 6\})) \approx \frac{8269}{100\,000}$, $H(C(\{1, 2, 3, 4, 5\})) \approx \frac{523}{6250}$, $H(C(\{1, 2, 3, 4, 5, 9\})) \approx \frac{8541}{100\,000}$, $H(C(\{1, 2, 3, 4, 5, 7\})) \approx \frac{1077}{12\,500}$, $H(C(\{1, 2, 3, 4, 5, 6\})) \approx \frac{2169}{25\,000}$, $H(C(\{1, 2, 3, 4, 5, 6, 8\})) \approx \frac{8851}{100\,000}$, $H(C(\{1, 2, 3, 4, 5, 6, 7\})) \approx \frac{8889}{100\,000}$, $H(C(R_8)) \approx \frac{1809}{20\,000}$, $H(C(R_9)) \approx \frac{2291}{25\,000}$, $H(C(R_{10})) \approx \frac{9257}{100\,000}$, $H(C(R_{13})) \approx \frac{1889}{20\,000}$, $H(C(R_{18})) \approx \frac{961}{10\,000}$, and $H(C(R_{34})) \approx \frac{49}{500}$.

ApproximationCoefficientDifferenceDistribution

Let ξ be an irrational number with regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n/B_n. Let

$$\Theta_n = B_n^2 \left|\xi - \frac{A_n}{B_n}\right|,$$

and $a \in \mathbb{Z}^+$. Then for almost all $\xi \in [0, a/(1+a)]$ with $b_n = a$ the density function for the distribution of $|\Theta_{n+1} - \Theta_{n-1}|$ is

$$p(z) = \frac{1}{\ln(2)} \left(\frac{1}{a} \ln\left(\frac{2+a}{a}\right) + \frac{1}{a} \ln\left(\frac{a-z}{a+z}\right)\right).$$

ApproximationCoefficientDistributions

Let $0 < \xi < 1$ be an irrational number with regular continued fraction representation

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

and A_n / B_n the sequence of its convergents. Let $\Theta_n(\xi)$ be the approximation coefficients

$$\Theta_n(\xi) = B_n^2 \left| \xi - \frac{A_n}{B_n} \right|.$$

Then, as $n \to \infty$, the following holds with respect to the Lebesgue measure λ on $[0, 1]$:

$$\lim_{n\to\infty} \lambda(\Theta_k(\xi) < t) = \begin{cases} \frac{t}{\ln(2)} & \text{for } 0 \le t \le 1/2 \\ \frac{1}{\ln(2)} (1 - t + \ln(2\,t)) & \text{for } 1/2 \le t \le 1 \end{cases}$$

$$\lim_{n\to\infty} \lambda(\Theta_{k-1}(\xi) < s \bigwedge \Theta_k(\xi) < t) = \begin{cases} \frac{1}{\ln(2)} \frac{1}{\sqrt{1-4\,s\,t}} & \text{for } 0 \le s \bigwedge 0 \le t \bigwedge s + t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{n\to\infty} \lambda\left(\frac{\frac{\Theta_{k+1}(\xi)}{B_{k+1}^2}}{\frac{\Theta_k(\xi)}{B_k^2}} < t \right) = \begin{cases} \frac{1}{\ln(2)} \left(\ln(t+1) - \frac{t \ln(t)}{t+1} \right) & \text{for } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{n\to\infty} \lambda\left(\frac{B_{k+1}}{B_k} \Theta_k(\xi) < t \right) = \begin{cases} 0 & \text{for } 0 \le t \le 1/2 \\ \frac{1}{\ln(2)} \ln\left(2\,t^t\,(1-t)^{1-t}\right) & \text{otherwise.} \end{cases}$$

ApproximationCoefficientsRecursion1

Let ξ be the the regular continued fraction

$$\xi = b_0 + \underset{j=1}{\overset{M}{\mathrm{K}}} \frac{1}{b_j}$$

with $M \le \infty$, convergents A_n / B_n, and approximation coefficients

$$\theta_n = B_n^2 \left| \xi - \frac{A_n}{B_n} \right|.$$

Then the following recursion relation holds for $n > 1$:

$$\theta_{n+1} = \theta_{n-1} + b_{n+1} \sqrt{1 - \theta_{n-1}\,\theta_n} - b_n^2\,\theta_n.$$

ApproximationCoefficientsRecursion2

Let ξ be the the regular continued fraction

$$\xi = b_0 + \underset{j=1}{\overset{M}{\mathrm{K}}} \frac{1}{b_j}$$

with $M \leq \infty$, convergents A_n / B_n, and approximation coefficients

$$\theta_n = B_n^2 \left| \xi - \frac{A_n}{B_n} \right|.$$

Then the following recursion relations hold for $n > 1$:

$$\theta_{n+1} = \theta_{n-1} + \sqrt{1 - 4\,\theta_{n-1}\,\theta_n} \left\lfloor \frac{\sqrt{1 - 4\,\theta_{n-1}\,\theta_n} + 1}{2\,\theta_n} \right\rfloor - \theta_n \left\lfloor \frac{\sqrt{1 - 4\,\theta_{n-1}\,\theta_n} + 1}{2\,\theta_n} \right\rfloor^2.$$

ApproximationCoefficientSum

Let ξ be an irrational number with regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

with convergents A_n / B_n. Let

$$\Theta_n = B_n^2 \left| \xi - \frac{A_n}{B_n} \right|.$$

Then for almost all $\xi \in \mathbb{R}$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=3}^{N} \left| \Theta_{n+1} - \Theta_{n-1} \right| = \frac{2\,\gamma + 1 - \ln(2\,\pi)}{2\ln(2)}.$$

ApproximationCoefficientSums

Let $0 < \xi < 1$ be an irrational number with regular continued fraction representation

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

and A_n / B_n the sequence of its convergents. Let $\Theta_n(\xi)$ be the approximation coefficients

$$\Theta_n(\xi) = B_n^2 \left| \xi - \frac{A_n}{B_n} \right|.$$

Then the following identities hold for almost all ξ:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Theta_k(\xi) = \frac{1}{4\ln(2)}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Theta_k(\xi)\,\Theta_{k+1}(\xi) = \frac{1}{6}\left(1 - \frac{1}{4\ln(2)}\right)$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{B_{k+1}}{B_k}\,\Theta_k(\xi) = \frac{1}{2} + \frac{1}{4\ln(2)}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{\frac{\Theta_{k+1}(\xi)}{B_{k+1}^2}}{\frac{\Theta_k(\xi)}{B_k^2}} = \frac{\pi^2}{12\ln(2)}.$$

ApproximationCoefficientTVSequenceDistribution

Let ξ be an irrational number with regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n. Let

$$t_n = \underset{k=n}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

and

$$v_n = \underset{k=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_{n+1-k}}.$$

For almost all x, the sequence $\{t_n, v_n\}$ is distributed according to the density function

$$\mu(t, v) = \frac{1}{\ln(2)} \frac{1}{(1 + t\,v)^2}.$$

AroianContinuedFraction

Let p and q be real numbers and

$$c_n = \begin{cases} 1 & \text{for } n = 0 \\ \frac{(p+s)(-p-q-s)}{(p+2s)(p+2s+1)} & \text{for } n = 2s+1 \\ \frac{s(q-s)}{(p+2s-1)(p+2s)} & \text{for } n = 2s \end{cases}$$

$$C_n = x\, c_n$$

and

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{C_n}.$$

Then

$$\frac{\mathrm{B}_x(p, q)}{\mathrm{B}(p, q)} = \frac{\xi\, x^p (1-x)^q\, \Gamma(p+q)}{\Gamma(p+1)\, \Gamma(q)}.$$

ArwinFormula

Given a real root μ to

$$0 = b_2\, \mu^2 + b_1\, \mu + b_0,$$

P_p a solution to

$$0 = \left(b_2\, P_p^2 + b_1 \left(-P_p\right) + b_0\right) \bmod Q_p,$$

and integers P_p, Q_p, Q_t, z_x, z_{x-1}, y_x, y_{x-1}, α, β, and γ satisfying

$$\gcd(\alpha, \beta) = 1$$

$$\gcd(z_x, Q_t) = 1$$

$$\frac{\mu + P_p}{Q_p} = \frac{y_x \left(\alpha\, \mu^2 + \beta\, \mu + \gamma\right) + y_{x-1}}{z_x \left(\alpha\, \mu^2 + \beta\, \mu + \gamma\right) + z_{x-1}}$$

$$|y_x\, z_{x-1} - y_{x-1}\, z_x| = 1,$$

and let

$$A_2 = \frac{b_2}{b_0}$$

$$A_3 = \frac{b_3}{b_0}.$$

Then

$$\left|\alpha^3\, A_3 + \alpha^2\, A_2\, \beta + \beta^6\right| = \left|Q_p\, Q_t\right|.$$

AssociatedContinuedFractionTo2SeriesForGoldenRatio

Set

$$T(x) = \sum_{i=1}^{\infty} 2^{\lfloor i\,x \rfloor}$$

and

$t_n = 2^{F_{n-2}}$.

Then $T(\phi)$ is a transcendental real and has as its regular continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{t_k}.$$

AsymptoticBehaviorForFunctionsOfPartialQuotients

Let $\epsilon > 0$ and suppose that g is a function which behaves asymptotically like $p^{1-\epsilon}$, i.e., $g(p) = O\left(p^{1-\epsilon}\right)$, i.e., $g(p)$. If $\xi = [0; b_1, b_2, \ldots]$ is a continued fraction, then

$$\lim_{K \to \infty} \frac{1}{K} \sum_{n=1}^{K} g(b_n) = \sum_{p=1}^{\infty} g(p) \log_2\left(\frac{(p+1)^2}{p(p+2)}\right).$$

In particular, if $g(p) = \delta_{p,q}$ for some q, then

$$f_q = \log_2\left(\frac{(q+1)^2}{q(q+2)}\right),$$

where $f_q = \lim_{K \to \infty} N_q(K)/K$ for $N_q(K)$ the number of times the digit q occurs in the first K terms of ξ.

AsymptoticBoundForDiscrepancyOfCertainContinuedFractionRelatedSequences

Consider the closed hypocycloid S of q cusps whose parameterized form is given by

$$S(t) = \begin{cases} x(t) = (\theta - 1)\, r \cos(t) + r \cos((\theta - 1)\, t) \\ y(t) = (\theta - 1)\, r \sin(t) + r \sin((\theta - 1)\, t) \end{cases}$$

for $0 < \theta = p/q < 1$ and let $\{S\}^t$ denote the trace of S on the interval $I_t = [0, 2\pi t\, p/q]$, that is, $\{S\}^t$ is the partially completed plot of S on I_t. Further, let $\omega = \left\{\text{frac}(n\,\theta)^t\right\}_{n=1}^{\lfloor t \rfloor}$ where $\text{frac}(n\,\theta)^t$ denotes the finite portion of the fractional part of $(n\,\theta)$ corresponding to $\{S\}^t$. Under this construction, if ξ_n^t is the continued fraction representation of $(n\,\theta)^t$ for $n = 1, 2, \ldots$ and if ξ_n^t has bounded partial quotients, then the discrepancy $D_N(\omega)$ satisfies the asymptotic expression $D_N(\omega) = O\left(N^{-1} \ln N\right)$. Moreover, if ξ_n^t has partial quotients bounded by some K, then

$$N\, D_N(\omega) \le 3 + \left(\frac{1}{\ln(\phi)} + \frac{K}{\ln(K+1)}\right) \ln(N).$$

AsymptoticConvergentBehaviorOfLimitPeriodicContinuedFractions

For a limit periodic continued fraction $\xi = \mathrm{K}(b_n/1) = [0; b_1, b_2, \ldots]$ with $|b_n - (-1/4)| \le \frac{1-\beta^2}{4\left(4 n^2 - 1\right)}$, $0 \le \beta \le 1$, $n = 1, 2, \ldots$,

$$\left|\frac{h_n}{h_n - \frac{1}{2}}\right| \le \frac{2\,n + 2 + \beta}{1_\beta}$$

for $n = 1, 2, \ldots$ where $h_n = -S_n^{-1}(\infty)$, $S_n(0) = A_n/B_n$ is the nth approximant of ξ, and approximant function $S_n(w) = \frac{A_n + A_{n-1}\, w}{B_n + B_{n-1}\, w}$.

AsymptoticDigitSumDistribution

Let the number $0 < x < 1$ have the regular continued fraction expansion

$$x = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

and let $S_r(x)$ be the digit sums of the truncated partial denominator sequences

$$S_r(x) = \underset{k=1}{\overset{r}{\mathrm{K}}} b_k.$$

Furthermore, let $\phi(\xi)$ be the stable distribution with density

$$\phi(\xi) = \mathrm{PDF}\left[\mathrm{StableDistribution}\left[0,\ 1,\ 1,\ \ln\left(\frac{\pi}{2}\right),\ \frac{\pi}{2}\right],\ \xi\right]$$

and μ the ordinary Lebesgue measure on the real line. Then

$$\lim_{r\to\infty} \sup_{z\in\mathbb{R}} \left(m(\{x : x \in (0,\ 1) \bigwedge S_r(z) \le z\}) - \int_{-\infty}^{z \ln(2)/r+\gamma-\ln(r/\ln(2))} \phi(\xi)\, d\xi \right).$$

AsymptoticDistributionOfCoefficientsForIrrationalContinuedFractions

Let $\xi = [0; b_1,\ b_2,\ \ldots\ ]$ be the continued fraction representation of an irrational number $\alpha \in (0,\ 1)$, let $N_p(K)$ be the number of times the digit p occurs in the first K terms of ξ, and let $f_p = \lim_{K\to\infty} N_p\,(K)/K$ if it exists. Then with probability 1, the coefficients b_j of ξ are distributed asymptotically and

$$f_p = \log_2\left(\frac{(p+1)^2}{p(p+2)}\right).$$

AsymptoticModularPropertiesOfDigits

Let $0 < \xi < 1$ be an irrational number with the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then for any $m \in \mathbb{Z}^+$, $1 \le j \le m-1$, the following identity holds for almost all ξ

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{k_k \bmod m,j} = \frac{1}{\ln(2)} \ln\left(\frac{\Gamma\left(\frac{j}{m}\right)\Gamma\left(\frac{j+2}{m}\right)}{\Gamma\left(\frac{j+1}{m}\right)^2}\right).$$

AsymptoticRelativeDigitFrequency

Let ξ be an irrational number with the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then for any $j \in \mathbb{Z}^+$, the following identity holds for almost all ξ

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{j,b_k} = \frac{1}{\ln(2)} \ln\left(1 + \frac{1}{j(j+2)}\right).$$

AsymptoticRelativeDigitFrequencyWithErrorTerm

Let ξ be an irrational number with the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then for any $j \in \mathbb{Z}^+$ and any $\varepsilon > 0$, the following identity holds for almost all ξ;

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \delta_{j,b_k} = \frac{1}{\ln(2)} \ln\left(1 + \frac{1}{j(j+2)}\right) + o\left(\sqrt{\frac{1}{n} \ln^3(n)}\right).$$

AsymptoticRelativeDigitRangeFrequency

Let ξ be an irrational number with the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then for any j_1, $j_2 \in \mathbb{Z}^+$ with $j_1 \le j_2$, the following identity holds for almost all ξ:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathrm{Boole}[j_1 \le k \le j_2] = \frac{1}{\ln(2)} \ln\left(\frac{(j_1+1)(j_2+1)}{j_1(j_2+2)}\right).$$

AsymptoticRelativeExceedingDigitFrequency

Let $0<\xi<1$ be an irrational number with the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then for any $j \in \mathbb{Z}^+$, the following identity holds for almost all ξ

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \text{Boole}[b_k > j] = \frac{1}{\ln(2)} \ln\left(1+\frac{1}{j}\right).$$

AsymptoticsForHausdorffDimensionForBoundedPartialQuotients

Let n be a natural number, E be a subset of the natural numbers less than or equal to n, $E(R)$ be the regular continued fractions ξ whose partial denominators lie in E, and H be the Hausdorff dimension. Then

$$H(E(R)) = 1 - \frac{6}{n\,\pi^2} - \frac{72\ln(n)}{n^2\,\pi^4} + O\left(\frac{1}{n^2}\right).$$

AuricTheorem

Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n}$$

be a generalized continued fraction where $a_n \neq 0$, and X_n be the three term recurrence solution continued fraction of ξ. Given $X_n \neq 0$ and

$$\prod_{n=0}^{\infty} \left(\frac{\prod_{m=1}^{n} -a_n}{X_{-1+n}\,X_n}\right) = \infty,$$

then ξ converges to

$$-\frac{X_0}{X_{-1}}.$$

AverageContinuedFractionLengthOfARational

Let q be an integer and for rational numbers $0 < p/q < 1$, $\gcd(p, q) = 1$ and let

$$\frac{p}{q} = \underset{k=1}{\overset{L\left(\frac{p}{q}\right)}{\mathrm{K}}} \frac{1}{b_k}$$

be its regular continued fraction expansion.

Then the following limit for the average length of a continued fraction of a proper fraction with denominator q holds:

$$\lim_{q\to\infty} \frac{1}{\phi(q)} \sum_{\substack{p=1\\ \gcd(p,q)=1}}^{q-1} L\left(\frac{p}{q}\right) = \frac{12\ln(2)}{\pi^2}\ln(q) + C_P + O\left(\frac{1}{q^{1/6+\varepsilon}}\right)$$

where $\varepsilon > 0$ and

$$C_P = \frac{12\ln(2)}{2\pi^2}(48\ln(A) - 2 - \ln(2) - 4\ln(\pi)) - \frac{1}{2}.$$

AverageGrowthOfHalfRegularContinuedFractionConvergentsDenominators

Let

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{\varepsilon_k}{\beta_k}$$

be a half-regular continued fraction expansion and A_n/B_n the sequence of its convergents. Here $-1/2 < \xi < 1/2$ and $\xi \notin \mathbb{Q}$ and $\varepsilon_k \in \{-1, 1\}$, $\beta_k \in \mathbb{Z}^+$, $\beta_k \geq 2$ and $\beta_k + \varepsilon_{k+1} \geq 2$, $\varepsilon_1 = \mathrm{sgn}(\xi)$, $|\beta_1 - 1/|\xi|| < 1/2$.

Then for almost all $-1/2 < \xi < 1/2$, the following holds:

$$\lim_{n\to\infty} \frac{\ln(B_n)}{n} = \frac{\pi^2}{\ln(\phi)}.$$

AverageOfIteratedGaussMap

Let τ be the Gauss map

$\tau : \mathbb{R} \to \mathbb{Z}$

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Then for any Borel subset A of the interval $[0, 1]$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} I_A(\tau^n) = \frac{1}{\ln(2)} \int_0^1 \frac{I_A(x)}{1+x}\, dx.$$

where $I_A(x)$ is the indicator function of the set A.

AverageOfIteratedHalfRegularGaussMap

Let τ be the Gauss map equivalent for half-regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{\varepsilon_k}{\beta_k}$$

where $-1/2 < \xi < 1/2$ and $\xi \notin \mathbb{Q}$ and $\varepsilon_k \in \{-1, 1\}$, $\beta_k \in \mathbb{Z}^+$, $\beta_k \geq 2$ and $\beta_k + \varepsilon_{k+1} \geq 2$, $\varepsilon_1 = \mathrm{sgn}(\xi)$, $|\beta_1 - 1/|\xi|| < 1/2$ defined as

$$\tau(\xi) = \underset{k=2}{\overset{\infty}{\mathrm{K}}} \frac{\varepsilon_k}{\beta_k}.$$

Then for every Lebesgue-measurable function f and for almost all $-1/2 < \xi < 1/2$ the following holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(\tau^n(\xi)) = \frac{1}{\ln(\phi)} \int_{-1/2}^{1/2} f(\sigma) \left(\begin{cases} \frac{1}{\phi+t} & \text{for } \sigma < 0 \\ \frac{1}{\phi+1+t} & \text{for } \sigma > 0 \end{cases} \right) d\sigma.$$

BadlyApproximableNumbersHavePoorRationalApproximations

Let ξ be a regular continued fraction, ϵ be a positive real, and x be a rational number p/q. Then $\exists_\epsilon \forall_x |-x + \xi| \geq \epsilon / q^2 \Leftrightarrow \xi$ is badly approximable.

BakerBoundForUniformConvergenceOfHolomorphicPadeApproximants

Let U be a disk, r be the disk radius of U, $f(z)$ be a formal power series that converges on U, $f_n(z)$ be the Padé approximants diagonal for f at 0, and V_n be the complex poles set for $f_n(z)$ in U. Then given $V_n = \emptyset$, the sequence f_n converges uniformly on U.

BankierGeneralizationOfGaloisTheoremOnPurePeriodContinuedFractions

Let ξ_1 be a continued fraction with periodic partial numerators and denominators

$$\xi_1 = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k}$$

$$a_{k+n} = a_n$$

$$b_{k+n} = b_n.$$

Let ξ_2 be the continued fraction with periodic partial numerators and denominators

$$\xi_2 = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_{n-k+1}}{b_{n-k}}$$

where ξ_1 and ξ_2 converge and $a_n \neq 0$, then

$$\xi_2 = -z_0$$

where z_0 is the conjugate of ξ_1 as quadratic expressions.

BaseComplementContinuedFractions

Let p_1/q_1 and p_2/q_2 be two rational number $(p_1, q_1, p_1, q_1 \in \mathbb{Z}^+)$ with regular continued fraction expansions

$$\frac{p_1}{q_1} = b_0^{(1)} + \underset{k=1}{\overset{n^{(1)}}{\mathrm{K}}} \frac{1}{b_k^{(1)}}$$

$$\frac{p_2}{q_2} = b_0^{(2)} + \underset{k=1}{\overset{n^{(2)}}{\mathrm{K}}} \frac{1}{b_k^{(2)}},$$

and let $A_n^{(1)} / B_n^{(1)}$ and $A_n^{(2)} / B_n^{(2)}$ be their convergents sequences. Define the fraction

$$\xi = \frac{A_{n^{(2)}-1}^{(2)} p_1 + p_2 q_1}{B_{n^{(2)}-1}^{(2)} p_1 + q_2 q_1}$$

with regular continued fraction expansion

$$\xi = b_0^{(\xi)} + \underset{k=1}{\overset{n^{(\xi)}}{\mathrm{K}}} \frac{1}{b_k^{(\xi)}}$$

and convergents $A_n^{(\xi)} / B_n^{(\xi)}$. Then the following identity holds:

$$\frac{A_{n^{(\xi)}-1}^{(\xi)}}{B_{n^{(\xi)}-1}^{(\xi)}} = \frac{A_{n^{(2)}-1}^{(2)} A_{n^{(1)}-1}^{(1)} + p_2 B_{n^{(1)}-1}^{(1)}}{B_{n^{(2)}-1}^{(2)} A_{n^{(1)}-1}^{(1)} + q_2 B_{n^{(1)}-1}^{(1)}}.$$

BasicPropertiesOfContinuants

The continuants $K_n(x_1, x_2, \ldots, x_n)$ have the following properties:

$$K_n(1, \ldots, 1) = F_{n+1}$$

$$K_n(x_1, \ldots, x_n + y) = K_n(x_1, \ldots, x_n) + y\, K_{n-1}(x_1, \ldots, x_{n-1})$$

$$K_n(x_1, \ldots, x_n)\, K_{n-2}(x_2, \ldots, x_{n-1}) - K_{n-1}(x_2, \ldots, x_n)\, K_n(x_1, \ldots, x_{n-1}) = (-1)^n$$

$$K_{m+n}(x_1, \ldots, x_{m+n})\, K_l(x_{m+1}, \ldots, x_{m+l}) - K_{m+l}(x_1, \ldots, x_{m+1})\, K_n(x_{m+1}, \ldots, x_{m+1}) = (-1)^n K_{m-1}(x_1, \ldots, x_{m-1})\, K_{n-l-1}(x_{m+l+2}, \ldots, x_{m+n})$$

If a real number ξ has the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k},$$

then

$$\xi = \frac{K_{n+1}(b_0, b_1, \ldots, b_n)}{K_n(b_1, \ldots, b_n)}.$$

BasicPropertiesOfRegularContinuedFractionConvergents

Let ξ be a real number with regular continued fraction expansion

$$\xi = b_0 + \mathop{\mathrm{K}}_{k=1}^{M} \frac{1}{b_k}.$$

(M possibly ∞ for irrational numbers) with convergents A_n/B_n. The convergents have the following properties:

Recurrences:

$A_n = b_n\, A_{n-1} - A_{n-2}$ where $A_{-1} = 1$ and $A_0 = b_0$
$B_n = b_n\, B_{n-1} - B_{n-2}$ where $B_{-1} = 0$ and $B_0 = 1$

Identities:

$$\frac{A_n}{B_n} - \frac{A_{n+1}}{B_{n+1}} = -\frac{(-1)^{n-1}}{B_{n-1}\, B_n}$$

$$\xi = b_0 + \sum_{n=0}^{\infty} \frac{1}{B_n\, B_{n+1}}$$

$$\xi = b_0 + \mathop{\mathrm{K}}_{k=1}^{M} \frac{1}{\delta_{M,k}\, \xi_M + \left(1 - \delta_{M,k}\right) b_k},$$

where

$$\xi_M = b_M + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{1}{b_{M+k}},$$

$\gcd(A_{n-1},\, A_n) = 1,$

and

$\gcd\,(B_{n-1},\, B_n) = 1.$

Bounds:

$A_n \leq F_n$ and $B_n \leq F_{n+1}$

$$\frac{A_{2\,n}}{B_{2\,n}} < \xi < \frac{A_{2\,n+1}}{B_{2\,n+1}}$$

$$\frac{1}{2\,B_n} < |\xi\, B_n - A_n| \leq \frac{1}{B_n}$$

$$\frac{A_n}{B_n} = \xi + (-1)^{n-1}\, \frac{|\delta|}{B_n^2},$$

where

$$\frac{1}{b_{n+1} + 2} < |\delta| < \frac{1}{b_{n+1}}.$$

Bounds on differences:

$$\left|\xi - \frac{A_n}{B_n}\right| > \left|\xi - \frac{A_{n+1}}{B_{n+1}}\right|$$

$$\frac{1}{B_n(B_{n+1} + B_n)} < \left|\xi - \frac{A_n}{B_n}\right| < \frac{1}{B_n\, B_{n+1}}$$

$$\left|\xi - \frac{A_n}{\ldots}\right| < \frac{1}{\ldots}$$

$$\left| \quad B_n \right| \quad 2 B_n^2$$

$$\left|\xi - \frac{A_n}{B_n}\right| < \frac{1}{F_{n+1} F_{n+2}}$$

$$\left|\xi - \frac{A_n}{B_n}\right| < \frac{1}{\phi^{2n-1}}$$

$$\left|\xi - \frac{A_n}{B_n}\right| < \frac{1}{\left(\sqrt{2}\right)^n}$$

$$\left|\xi - \frac{A_n}{B_n}\right| \leq \left|\xi - \frac{A}{B}\right|$$

for all $A \in \mathbb{Z}^+$, $B \in \mathbb{Z}^+$ and $0 \leq B \leq B_n$.

BauerMuirTransformation

Given a sequence $w = \{w_n\}$ of complex numbers, the Bauer-Muir transformation of a generalized continued fraction ξ of the form

$$\xi = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}}$$

with respect to w is the continued fraction ζ of the form

$$\zeta = d_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{c_m}{d_m}$$

whose canonical numerators C_n, respectively canonical denominators D_n, are defined by the recursion relations $C_{-1} = 1$, $C_n = A_n + w_n A_{n-1}$, $D_{-1} = 0$, and $D_n = B_n + w_n B_{n-1}$ for $n = 1, 2, 3, \ldots$. Here, A_n / B_n denotes the canonical nth convergents of ξ.

One well-know result concerning the Bauer-Muir transformation is a characterization of its existence. In particular, given a generalized continued fraction ξ of the form stated above and a corresponding complex sequence $w = \{w_n\}$, the Bauer-Muir transformation of ξ with respect to w exists if and only if $\lambda_n \neq 0$ where here,

$$\lambda_n = a_n - w_n (b_n + w_n)$$

for $n = 1, 2, 3, \ldots$. Moreover, Lorentzen and Waadeland showed that if it exists, the Bauer-Muir transformation of ξ with respect to w has the form

$$\zeta = b_0 + w_0 + \frac{\lambda_1}{b_1 + w_1 + \frac{c_2}{d_2 + \frac{c_3}{d_3 + \cdots}}}$$

where $c_n = a_{n-1} q_{n-1}$ and $d_n = b_n + w_n - w_{n-2} q_{n-1}$ for $q_n = \lambda_{n+1} / \lambda_n$, $n = 1, 2, 3, \ldots$. More specific properties of the Bauer-Muir transformation have also been studied in relation to various other topics including but not limited to the Rogers-Ramanujan continued fraction.

BestLeftApproximation

Let α be an irrational number in (0, 1). Then a fraction p/q is called a best left approximation to α if (i) $p/q<\alpha$ and (ii) there is no fraction $x/y \in (p/q, \alpha)$ with a denominator $y \leq q$.

BestRationalApproximation

A fraction p/q is called a best rational approximation of the real number ξ if

$$\left|\xi-\frac{p}{q}\right|<\left|\xi-\frac{r}{s}\right|$$

for any integers r and s such that $s \leq q$ and $p/q \neq r/s$.

Let ξ have the regular continued fraction expansion

$$\xi=b_0+\underset{k=1}{\overset{M}{\mathrm{K}}}\frac{1}{b_k}$$

(for M possibly ∞) with convergents A_n/B_n.

Then every convergent A_n/B_n is best rational approximation of ξ.

BestRationalApproximationTheorem

Let

$$\xi=b_0+\underset{n=1}{\overset{N}{\mathrm{K}}}\frac{1}{b_n}$$

be a regular continued fraction with value ξ and convergents A_n/B_n, and let p and q be two positive integers such that

$$\left|\xi-\frac{p}{q}\right| \leq \left|\xi-\frac{A_n}{B_n}\right|.$$

Then $q \geq B_n$. Moreover, if $q=B_n$, then $p=A_n$.

BestRightApproximation

Let α be an irrational number in $(0, 1)$. Then a fraction p/q is called a best right approximation to α if (i) $p/q > \alpha$ and (ii) there is no fraction $x/y \in (\alpha, p/q)$ with a denominator $y \geq q$.

BijectionFromPowerSetOfNaturalNumbersToPositiveRealsViaContinuedFractions

Define f to be the function from the powerset of the natural numbers to the nonnegative real numbers by

$$f(A) = \begin{cases} 0 & \text{for } A = \emptyset \\ n & \text{for } A = \{n\} \\ a_1 + \underset{k=1}{\overset{m}{\mathrm{K}}} \frac{1}{a_{k+1} - a_k + \delta(k-m)} & \text{for } |A| = m \\ a_1 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{a_{k+1} - a_k} & \text{for } |A| = \infty. \end{cases}$$

Then f is a bijection between the powerset of the natural numbers and the nonnegative real numbers.

BinaryQuadraticFormRepresentationOfNegative1

Let D be a positive integer that is not a perfect square, let $x^2 - D\,y^2$ represent -1, let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction expansion of $\sqrt{D}$, and let P_n/Q_n be the nth complete quotient of ξ. Then

$$D = Q_n^2 + P_n^2,$$

where Q_n is odd, and

$$\gcd(P_n,\ Q_n) = 1.$$

BinaryQuadraticFormRepresentationOfPlusOrMinusb

Let D be a positive integer that is not a perfect square, let $x^2 - D\,y^2$ represent -1, let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction expansion of $\sqrt{D}$, let P_n/Q_n be the n^{th} complete quotient of ξ, and let A_n/B_n be the n^{th} convergent. If

$$(T_1,\ U_1) = (A_{n-1} - A_{n-2},\ B_{n-1} - B_{n-2})$$

then

$$T_1^2 - D\,U_1^2 = (-1)^n\,2\,P_n.$$

Similarly, if

$$(T_2,\ U_2) = (A_{n-1} + A_{n-2},\ B_{n-1} + B_{n-2})$$

then

$$T_2^2 - D\,U_2^2 = (-1)^{n-1}\,2\,P_n.$$

Finally,

$$\gcd(T_1,\ U_1) = \gcd(T_2,\ U_2) = 1.$$

BlockComplexityAsymptoticForContinuedFractionsOfAlgebraics

Let α be an algebraic number where $0 < \alpha < 1$,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction of α, and $p(n,\, b_n)$ be its block complexity. Then given that b_n is not ultimately periodic, it follows that $\lim_{n\to\infty} p(n,\, b_n)/n = \infty$.

BlockComplexityBoundForContinuedFractionsOfAlgebraics

Let α be an algebraic number where $0 < \alpha < 1$,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction of α, and $p(n,\, b_n)$ be its block complexity. Then given that b_n is not ultimately periodic, it follows that $p(n,\, b_n) \geq n + 1$.

BohmerFormula

Given a regular continued fraction

$$\alpha = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

with convergents A_n / B_n and an integer $c > 1$, then the continued fraction for the approximant function

$$S_b(\alpha) = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{t_n}$$

is given by

$$t_n = \begin{cases} b_0\, c & \text{for } n = 0 \\ \frac{c^{B_n} - c^{B_{n-2}}}{c^{B_{n-1}} - 1} & \text{otherwise.} \end{cases}$$

BoundedBranchedFractionsWithNaturalElementsConverge

Any bounded branched fraction with natural elements converges.

BoundedPartialQuotientsForContinuedFractionForBaumSeries

Let $K = F(x^{-1}) = F_2(x^{-1})$ be the formal power series in $1/x$ with coefficients in the field of two elements. Given f in K with ξ its regular continued fraction and b_n its partial denominators where

$$f^3 + \frac{f}{x} + 1 = 0,$$

then $\deg(b_n) \leq 2$.

BoundsOfErrorSumFunctionsOfContinuedFractions

Let α be an irrational number where $0 \leq \alpha \leq 1$, ξ be the regular continued fraction of α, $\mathcal{E}(\alpha)$ be the absolute error sum function of ξ, and $\mathcal{E}^*(\alpha)$ be the error sum function of ξ. Then $\mathcal{E}(\alpha) \leq \phi$ and $\mathcal{E}^*(\alpha) \leq 1$.

BoundsOnContinuedFractionApproximants

Let $\alpha \in \mathbb{R}$ be an arbitrary real number with associated continued fraction ξ and let P_n / Q_n denote the nth convergent of ξ for $n = 1, 2, \ldots$. Then

$$\left|\alpha - \frac{P_n}{Q_n}\right| < \frac{1}{Q_n\, Q_{n+1}} < \frac{1}{Q_n^2}$$

for all n.

BranchedContinuedFraction

A branched continued fraction is an expression of the form

$$\xi = b_0 + \sum_{i_1=1}^{N} \frac{a_{i_1}}{b_{i_1} + \sum_{i_2=1}^{N} \frac{a_{i_1,i_2}}{n_{i_1,i_2} + \sum_{i_3=1}^{N} \frac{a_{i_1,i_2,i_3}}{b_{i_1,i_2,i_3} + \ldots}}}.$$

BranchedContinuedFraction:BoundedBranching

A branched continued fraction X of the form

$$X = b + \sum_{i_1=1}^{N} \frac{a_{i_1}}{b_{i_1} + \sum_{i_2=1}^{N_{i_1}} \frac{a_{i_1\, i_2}}{b_{i_1\, i_2\, i_3} + \sum_{i_3=1}^{N_{i_1\, i_2}} \frac{a_{i_1\, i_2\, i_3}}{\ldots}}}$$

is said to have bounded branching if the branching numbers N, $N_{i_1\, i_2\, \ldots\, i_k}$ of X are all bounded by one number.

BranchedContinuedFraction:BranchedFractionWithNaturalElements

Given a branched continued fraction X of the form

$$X = b + \sum_{i_1=1}^{N} \frac{a_{i_1}}{b_{i_1} + \sum_{i_2=1}^{N_{i_1}} \frac{a_{i_1\, i_2}}{b_{i_1\, i_2\, i_3} + \sum_{i_3=1}^{N_{i_1\, i_2}} \frac{a_{i_1\, i_2\, i_3}}{\ldots}}},$$

the numbers $a_{i_1\, i_2\, \ldots\, i_k}$, b, and $b_{i_1\, i_2\, \ldots\, i_k}$ are called the elements of X. If all elements except for possibly b are natural numbers, X is said to be a branched fraction with natural elements.

BranchedContinuedFraction:Convergence

For any branching fraction X of the form

$$X = b + \sum_{i_1=1}^{N} \frac{a_{i_1}}{b_{i_1} + \sum_{i_2=1}^{N_{i_1}} \frac{a_{i_1 i_2}}{b_{i_1 i_2 i_3} + \sum_{i_3=1}^{N_{i_1 i_2}} \frac{a_{i_1 i_2 i_3}}{\dots}}},$$

one can construct so-called convergent fractions X_m of the form

$$X_m = b + \sum_{i_1=1}^{N} \frac{a_{i_1}}{b_{i_1} + \sum_{i_2=1}^{N_{i_1}} \frac{a_{i_1 i_2}}{\ddots \, b_{i_1 i_2 \dots i_{m-1}} + \sum_{i_m=1}^{N_{i_1 i_2 \dots i_{m-1}}} \frac{a_{i_1 i_2 \dots i_m}}{b_{i_1 i_2 \dots i_m}}}}$$

by removing all elements from X with indices greater than or equal to $m+1$ for $m = 1, 2, \dots$. If the limit of X_m exists as $m \to \infty$ and if $\alpha = \lim_{m\to\infty} X_m$, then it is said that X converges and represents α.

BranchedContinuedFraction:PeriodicBranchedFraction

Two branching fractions are said to be graphically equal if their branching numbers are the same and if the elements with equal indices coincide; branching fractions which are not graphically equal are said to be graphically different. A branching continued fraction X of the form

$$X = b + \sum_{i_1=1}^{N} \frac{a_{i_1}}{b_{i_1} + \sum_{i_2=1}^{N_{i_1}} \frac{a_{i_1 i_2}}{b_{i_1 i_2 i_3} + \sum_{i_3=1}^{N_{i_1 i_2}} \frac{a_{i_1 i_2 i_3}}{\dots}}}$$

is said to be periodic if it contains a finite number of pairwise graphically different subfractions.

BrodenBorelLevyTheorem

Let τ be the Gauss map

$\tau : \mathbb{R} \to \mathbb{Z}$

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

and let $0 < \xi < 1$ have the regular continued fraction expansion

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

The Lebesgue measure λ of all ξ in [0, 1] that have the initial partial denomina-tors $b_1, b_2, \ldots, b_n$ and property that $\tau^n(\xi) < \xi$ is

$$\lambda = \frac{(s_n + 1)\,\xi}{s_n\,\xi + 1},$$

where

$$s_n = \underset{k=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_{n-k+1}}.$$

BundschuhSumExpansion

Let

$$\alpha_g(\beta) = \sum_{k=1}^{\infty} \frac{g-1}{g^{\lfloor k\,\beta \rfloor}}$$

where $g \in \mathbb{Z}$ and $g > 1$ and β is an irrational number. Further, let

$$\frac{1}{\beta} = b_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

and A_n / B_n be its convergents. After defining a sequence C_k through

$$C_0 = b_0$$

$$C_n = g^{q(n-2)} \sum_{j=0}^{b_n-1} g^{j\,q(n-1)} \text{ where } n \geq 1$$

the following identity holds:

$$\alpha_g(\beta) = C_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{C_n}.$$

BuslaevCounterexampleToHolomorphicPadeConjecture

Let

$$\zeta = \frac{1}{2}\left(-1 - i\sqrt{3}\right)$$

and

$$f(z) = \frac{3(\zeta+9)z^3 + 6z^2 + \sqrt{4z^6 + 81\left(3 - (\zeta+3)z^3\right)^2} - 27}{2z\left((\zeta+9)z^2 + 9z + 9\right)}$$

be a hyperelliptic function set, and $f_1(z)$ be the holomorphic function thats is the branch with $f_1(0) = 0$. Then it is not the case that $f_1(z)$ satisfies the Padé conjecture.

BuslaevCriteriaForContinuedFractionConvergence

Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n}$$

be a generalized continued fraction. Then given $\limsup_{n\to\infty} |-1 + b_n| + 2 \limsup_{n\to\infty} \sqrt{|-1 + a_n + b_n|} < 1$, the continued fraction converges.

CantorSetEqualitiesForRealNumbersWhoseContinuedFractionsHavePartialQuotientsLessThanOrEqualTo2Or3

Let F_k be the real numbers whose regular continued fractions have partial quotients less than or equal to k and G_k be the interval containing it.

$G_k = [\min(F_k), \max(F_k)]$

Then

$$3F_3 = 3G_3 = \left[\frac{1}{2}\left(\sqrt{21} - 3\right), \frac{3}{2}\left(\sqrt{21} - 3\right)\right]$$

and

$$4F_2 = 4G_2 = \left[2\left(\sqrt{3} - 1\right), 4\left(\sqrt{3} - 1\right)\right].$$

CDuallyReducedIrrationalNumber

In irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ with conjugate α' is C-dually reduced if $\alpha > 1$ and $\alpha' < 0$.

CDuallyRegularFractionsConvergeToIrrationals

Any C-dually regular continued fraction ξ converges to some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

CentralLimitTheoremForContinuedFractionConvergenceOfDecimalApproximations

Let x be an irrational number where $0 < x < 1$ and

$d_n(x) = 10^{-n} \lfloor 10^n x \rfloor$

$e_n(x) = 10^{-n} (\lfloor 10^n x \rfloor + 1)$

be decimal approximations of x. Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction of x,

$$d_n(x) = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n^{(d)}}$$

be the regular continued fraction of $d_n(x)$,

$$e_n(x) = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n^{(e)}}$$

be the regular continued fraction of $e_n(x)$, and

$$k_n(x) = \sup_i \left\{ i : \forall\, i \le n \bigwedge b_i^{(d)} = b_i^{(e)} \right\}.$$

Let S be irrational numbers x with $(k_n(x) - a\, n) \big/ \left(\sqrt{n}\ \sigma\right) \le z$, where

$$a = \frac{6 \ln(2) \ln(10)}{\pi^2}$$

and σ is a positive constant. Then

$\forall\, i < k_n(x),\ b_i = b_i^{(d)} = b_i^{(e)}$

and

$$\lim_{n \to \infty} (m(S)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt,$$

where m is the Lebesgue measure.

CFractionForCertainPowerSeries1

The power series $P(x) = c_0 + \sum_{i=1}^{\infty} c_i\, x^{\lambda_i\, 2^{i-1}}$, $c_i \neq 0$, $i = 0, 1, 2, \ldots$, $\lambda_1 \geq 1$, has the corresponding continued fraction

$$\xi = c_0 + \cfrac{b_1\, x^{\lambda_1}}{1 + \cfrac{b_2\, x^{\lambda_1}}{1 + \cfrac{b_3\, x^{\lambda_1}}{1 + \cdots}}}$$

where $b_1, b_2, \ldots$ are given in terms of $c_1, c_2, \ldots$ by the formulas: $b_1 = c_1$, $b_{2\,n} = -a_{2\,n+1}$, $b_{2^i+1} = -c_1\, c_{i+1} \big/ c_i^2$, and $b_{n\,2^{i+1}+2^i+1} = (-1)^n\, b_{2^i+1}$ for $i, n = 1, 2, 3, \ldots$.

CFractionForCertainPowerSeries2

Under the hypothesis $\lambda_{i+1} \geq 2\,\lambda_i$ for $i = 1, 2, 3, \ldots$, the power series $P(x) = c_0 + \sum_{i=1}^{\infty} c_i\, x^{\lambda_i\, 2^{i-1}}$, $c_i \neq 0$, $i = 0, 1, 2, \ldots$, $\lambda_1 \geq 1$, has the corresponding continued fraction

$$\xi = c_0 + \cfrac{b_1\, x^{\alpha_1}}{1 + \cfrac{b_2\, x^{\alpha_2}}{1 + \cfrac{b_3\, x^{\alpha_3}}{1 + \cdots}}}$$

where the b_i are independent of the λ_i and are given in terms of $c_1, c_2, \ldots$ by the formulas $a_1 = c_1$, $a_{2\,n} = -a_{2\,n+1}$, $a_{2^i+1} = -c_1\, c_{i+1} \big/ c_i^2$, and $a_{n\,2^{i+1}+2^{i+1}} = (-1)^n\, a_{2^i+1}$ for $i, n = 1, 2, 3, \ldots$, and where the α_i are independent of the c_i and are given by the formulas: $\alpha_1 = \lambda_1$, $\alpha_{2\,n} = \alpha_{2\,n+1}$, $\alpha_{2^i+1} = \lambda_1 + \lambda_{i+1} - 2\,\lambda_i$, and $\alpha_{n\,2^{i+1}+2^i+1} = (-1)^n\, \alpha_{2^i+1}$ for $i, n = 1, 2, 3, \ldots$.

CFractionForCertainPowerSeries3

Given a formal power series $f_0(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$, there is an associated contin-ued fraction ξ_0 of the form

$$\xi_0 = \frac{c_0}{z - \alpha_0 - \frac{\beta_0}{z-\alpha_1 - \frac{\beta_1}{z-\alpha_2-\cdots-\frac{\beta_{r-2}}{z-\alpha_{r-1}-\cdots}}}}$$

where the partial numerators $A_k(z)$ and denominators $B_k(z)$ of ξ_0 are polynomi-als of the form

$$A_k(z) = \sum_{n=0}^{k-1} \zeta_{k,n} z^n$$

and

$$B_k(z) = \sum_{n=0}^{k} \zeta'_{k,n} z^n,$$

respectively, for some complex constants $\zeta_{k,n}$, $\zeta'_{k,n}$.

CFractionsInOneToOneCorrespondenceWithNonRationalPowerSeries

There is a one-to-one correspondence between corresponding type continued fractions

$$\xi = 1 + \frac{b_1 x^{\alpha_1}}{1 + \frac{b_2 x^{\alpha_2}}{1+\frac{b_3 x^{\alpha_3}}{1+\cdots}}}$$

and power series of the form

$$P(x) = 1 + \sum_{k=1}^{\infty} c_k x^k$$

which do not represent rational functions of x. Moreover, if the nth convergent of ξ is denoted $A_n(x)/B_n(x)$, this correspondence is completely determined by the recursion

$$B_n(x) P(x) - A_n(x) = (x^{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}})$$

where (x^s) denotes a formal power series in which the sth power is the smallest power of x which appears.

CharacterizationOfBestLeftRightFit

For any irrational number α in (0, 1), the Farey process zeroed in on α gives a sequence of best left and right approximations to α. Moreover, every best left and right approximation arises in this way.

CharacterizationOfContinuedFractionApproximants

For any real number x, let ξ be the continued fraction representation of x and let a_r be the $(r+1)$th partial quotient in ξ, i.e., $A_r/B_r = [b_0; b_1, b_2, \ldots, b_r]$ where A_r/B_r denotes the $(r+1)$th convergent of ξ. Then either (i) there are an infinite number of rational approximations p/q to x for which $q\,|q\,x - p| < \frac{1}{\sqrt{r^2+4}}$ or (ii) there exists an integer n_0 for which $a_n < r - 1$ for all $n \geq n_0$.

CharacterizationOfFareyIntervalsAndMediants

Let $[a/b, c/d]$ be a Farey interval. The two subintervals $[a/b, M]$ and $[M, c/d]$ formed by inserting the mediant $M = (a+c)/(b+d)$ are also Farey intervals and, among all rational numbers x/y such that $a/b < x/y < c/d$, M is the unique rational number with the smallest denominator (when reduced).

ChordalMetricOnRiemannSphere

Let w_1 and w_2 be two points in $\hat{\mathbb{C}}$, then the Euclidean length of the chord connecting the two points, known as the chordal distance or chrodal metric, is given by

$$S(w_1, w_2) = \begin{cases} \frac{2\,|w_1 - w_2|}{\sqrt{1+|w_1|^2}\,\sqrt{1+|w_2|^2}} & \text{for } (w_1, w_2) \in \mathbb{C}^2 \\ \frac{2}{\sqrt{1+|w_1|^2}} & w_1 \in \mathbb{C} \land w_2 = \tilde{\infty} \\ 0 & w_1 = w_2 = \tilde{\infty}. \end{cases}$$

CircularConvergenceTheorem

Let V_n be a region in the complex plane characterized by the fact that $v_n \in V_n$ if and only if $\mathrm{Re}\left(v_n\, e^{-i\alpha}\right) \geq -g_n \cos\alpha$ where the g_n are constants, $0 < g_n < 1$ for $n = 1, 2, \ldots$, and where $\alpha \in (-\pi/2, \pi/2)$. Let ξ be a continued fraction of the form $\xi = [0; b_1, b_2, \ldots]$ and denote by K_n the circular region $K_n = S_n(V_{n+1})$ of radius R_n, where $S_n(0) = A_n/B_n$ is the nth approximant of ξ and where $S_n(w) = \frac{A_n + A_{n-1}\, w}{B_n + B_{n-1}\, w}$ is the approximant function for all complex w. If d_n denotes the quotient

$$d_n = \frac{\prod_{\nu=1}^{n}\left(\frac{1}{g_{\nu+1}}\right)}{\sum_{k=0}^{n-1}\prod_{\nu=1}^{k}\left(\frac{1}{g_{\nu+1}} - 1\right)}$$

and if the sum

$$\sum_{\nu=2}^{\infty}\frac{d_{\nu-1}\, g_\nu(1 - g_{n+1})}{|b_\nu|}$$

diverges, then ξ converges to some complex number b.

ComparisonOfContinuedFractionPeriodsForRootDAndHalfOfOnePlusRootD1

Let D be a square free positive integer and for the regular continued fraction for $\sqrt{D}$, $b_1(n)$ its partial denominators, and $l_1(D)$ the period of $b_1(n)$ and for the regular continued fraction of $\left(\sqrt{D} + 1\right)/2$, $b_2(n)$ its partial denominators and $l_2(D)$ the period of $b_2(n)$. Given $\exists_{\text{odd } T \text{ and } U}\, T \wedge 2 - D * U \wedge 2 = 4$ then $l_2(D) + 4 \leq l_1(D) \leq 5\, l_2(D)$.

ComparisonOfContinuedFractionPeriodsForRootDAndHalfOfOnePlusRootD2

There are infinitely many D that are square free positive integers with $\mathrm{mod}(D, 4) = 1$ and for the regular continued fraction for $\sqrt{D}$, $b_1(n)$ its partial denominators, and $l_1(D)$ the period of $b_1(n)$ and for the regular continued fraction of $\frac{1}{2}\left(\sqrt{D} + 1\right)$, $b_2(n)$ its partial denominators and $l_2(D)$ the period of $b_2(n)$, it is the case that $l_1(D) = 3\, l_2(D) - 8$ and it is not true that $\exists_{\text{odd } T \text{ and } U}\, T^2 - \mathrm{DU}^2 = 4$ and it is not true that $\exists_{V \text{ and } W}\, V^2 - D\, W^2 = -1$ and $l_1(D)$ is unbounded.

ComplexRegularSymmetricPeriodicContinuedFractionsForImaginaryQuadraticIrrationals

Let D be a natural number, Q_0 be a positive integer,

$$x = \frac{\sqrt{D}}{Q_0}$$

be an irrational number, b_n be a natural number,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{a_n}$$

be the regular continued fraction of x, $l(d)$ be the regular continued fraction period of ξ, and

$$a_n = \begin{cases} i\, b_0 & \text{for } n = 0 \\ -i\, b_{n \bmod p} & \text{for } n \bmod p \neq 0 \\ -2\, i\, b_0 & \text{for } n \bmod p = 0 \end{cases}$$

be the partial denominator of ξ. Then b_n can be determined by also determining the sequences P_n and Q_n:

$$P_0 = 0$$

$$Q_{-1} = \frac{D - P_0^2}{Q_0}$$

$$b_n = \left\lfloor \frac{\sqrt{D}}{Q_n} + P_n \right\rfloor$$

$$P_{n+1} = b_n\, Q_n - P_n$$

$$Q_{n+1} = b_n\, (P_{n+1} - P_n) + Q_{n-1}.$$

ConditionalProbabilityTheoremForContinuedFractionCoefficients

Let $\xi = [0; b_1, b_2, \ldots]$ be an arbitrary continued fraction and suppose that $k > \ell$ are two positive integers. The conditional probability $\Pr\{b_k = p \mid b_\ell = q\}$ differs little from the unconditional probability $\Pr\{b_k = p\}$ which is asymptotic to f_p where $f_p = \lim_{K\to\infty} N_p(K)/K$ for $N_p(K)$ the number of times the digit p occurs in the first K quotients of ξ. More precisely, given an arbitrary constant β,

$$\Pr\{b_k = p \mid b_\ell = q\} - f_p = O\left(e^{-\beta \sqrt{k-\ell}}\right)$$

as the difference $k - \ell$ tends to infinity.

ContinuedFraction

The term "continued fraction" can be applied in several different contexts. In general, any expression ξ of the form

$$\xi = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

with terms a_j, b_k, $j = 1, 2, 3, \ldots$, $k = 0, 1, 2, \ldots$, consisting of arbitrary mathematical objects such as vectors in $\mathbb{C}^n$, $\mathbb{C}$-valued square matrices, Hilbert space operators, multivariate expressions, other such fractions, etc., is a continued fraction. Such expressions can terminate after finitely many terms or can continued infinitely. The terms a_k, respectively b_k, are called the partial numerators, respectively partial denominators, of ξ, and together, objects of the collection $\{a_k, b_k\}$ are called the elements of ξ.

Most typically, the term "continued fraction" is used to describe the scenario where a_j and b_k, $j = 1, 2, 3 \ldots$, $k = 0, 1, 2, \ldots$, are integers. In this case, any continued fraction which terminates after a finite number of terms defines a rational number $q \in \mathbb{Q}$. Otherwise, there are two distinct possibilities for the expression ξ which are characterized by the behavior of the rational numbers q_n defined by the finite expressions ξ_n of the form

$$\xi_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{\ddots + \cfrac{a_n}{b_n}}},$$

called the nth convergent of ξ. In particular, it may be the case that for some real number $\alpha \in \mathbb{R}$, $\xi_n \to \alpha$ as $n \to \infty$ whereby it is said that ξ is the continued fraction associated to α and that ξ_n converges to α; it is also possible, however, that ξ_n diverges as a sequence of rational numbers.

The above definition can be made both more general and more mathematically rigorous by way of the following function-theoretic construction. Given an ordered pair of sequences $(\{a_m\}, \{b_m\})$, a_m, $b_m \in \mathbb{C}$, $m \in \mathbb{Z}^+$, $a_m \neq 0$ for $m \geq 1$, one may consider the associated sequences $\{s_n(w)\}$, $\{S_n(w)\}$, $n = 0, 1, 2, \ldots$, of linear fractional transformations defined recursively by $s_0(w) = b_0 + w$,

$$s_n(w) = \frac{a_n}{b_n + w},$$

$S_0(w) = s_0(w)$, $S_n(w) = S_{n-1}(s_n(w))$ for $n = 1, 2, 3, \ldots$. By then defining the sequence $\{f_n\}$ so that, for each $n = 0, 1, 2, \ldots$, $f_n = S_n(0) \in \mathbb{C} \bigcup \{\infty\}$, one can define a continued fraction (with complex elements) to be the ordered pair $((\{a_m\}, \{b_m\}), \{f_n\})$.

ContinuedFraction:AlphaFraction

Let ξ be a real number. Then the α-continued fraction expansion for $1/2 \leq \alpha \leq 1$

$$\xi = \varepsilon_0 b_0 + \underset{j=1}{\overset{N}{\mathrm{K}}} \frac{\varepsilon_j}{b_j}$$

(where N is possibly infinity), $\varepsilon_j \in \{-1, 1\}$, and $b_j \in \mathbb{Z}^+$ can be calculated through the repeated application of the map τ_α: $[\alpha - 1, 1] \to [\alpha - 1, 1]$

$$\tau_\alpha(x) = \begin{cases} \frac{\operatorname{sgn}(x)}{x} - \left\lfloor \frac{\operatorname{sgn}(x)}{x} + 1 - \alpha \right\rfloor & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

ContinuedFraction:AlphaRosenFraction

Let ξ be a real number. Then the α-Rosen continued fraction expansion for $q \in \mathbb{Z}^+$, $q \geq 3$ and $\lambda_q = 2\cos(\pi/q)$ $0 \leq \alpha \leq 1/\lambda$

$$\xi = \varepsilon_0 b_0 + \underset{j=1}{\overset{N}{\mathrm{K}}} \frac{\varepsilon_j}{b_j}$$

(where N is possibly infinity), $\varepsilon_j \in \{-1, 1\}$, and $b_j \in \mathbb{Z}^+$ can be calculated through the repeated application of the map τ_α: $[(\alpha - 1)\lambda, \alpha\lambda] \to [(\alpha - 1)\lambda, \alpha\lambda]$

$$\tau_\alpha(x) = \begin{cases} \frac{\operatorname{sgn}(x)}{x} - \lambda \left\lfloor \frac{\operatorname{sgn}(x)}{\lambda x} + 1 - \alpha \right\rfloor & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

ContinuedFraction:AlternatingPositiveTermFraction

A Thron fraction ξ of the form

$$\xi = \cfrac{F_1 z}{1 + G_1 z + \cfrac{z F_2}{1 + G_2 z + \cfrac{z F_3}{1 + G_3 z + \cdots}}}$$

is said to be an alternating positive term fraction or APT-fraction if $F_m, G_m \in \mathbb{R} \setminus \{0\}$ satisfy the conditions $F_{2m-1} F_{2m} > 0$, $F_{2m-1}/G_{2m-1} > 0$ for $m = 1, 2, 3, \ldots$.

ContinuedFraction:ApproximantFunction

Given an ordered pair $(\{a_m\}_{m\in\mathbb{Z}^+}, \{b_m\}_{m\in\mathbb{Z}^*})$ of complex sequences with $a_m \neq 0$ for $m \geq 1$, the so-called nth continued fraction approximant function S_n is the function defined recursively for complex numbers $w \in \mathbb{C}$ by $S_0(w) = s_0(w)$ and $S_n(w) = S_{n-1}(s_n(w))$,

where $s_0(w) = b_0 + w$ and $s_n(w) = a_n (b_n + w)^{-1}$ for $n = 1, 2, 3, \ldots$. By way of a simple substitution for $n \geq 1$, it follows that S_n has the form

$$S_n(w) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}.$$

Despite nomenclature which has yet to be standardized, S_n is called the nth approximant function by authors as a way of acknowledging that $S_n(0)$ is the finite generalized continued fraction ξ_n of the form

$$S_n(0) = \xi_n = b_0 + \underset{m=1}{\overset{n}{\mathrm{K}}} \frac{a_m}{b_m},$$

also known as the nth approximant (or nth convergent) of the related infinite generalized continued fraction

$$\xi = b_0 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{a_m}{b_m}.$$

Though often unnamed, at least one other name related to S_n can be found in the literature. Cuyt et al. refer to $S_n(w_n) \in \hat{\mathbb{C}}$ as the nth modified approximant related to a sequence $w_0, w_1, w_2, \ldots$ of complex numbers. This term appears to be an acknowledgment of work done by Thron on the results of "modifying" the sequence $\{S_n(0)\}$ of continued fraction approximants to a sequence $\{S_n(w_n)\}$ for some prescribed complex $\{w_n\}$. Similar references and conventions can be found in the works by Lorentzen & Waadeland.

ContinuedFraction:ApproximationProperty

A number field $\mathbb{F}$ is said to have the approximation property if for every "irrational" α (i.e., $\alpha \notin \mathbb{F}$),

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{k\,Q^2}$$

for infinitely many rational elements $P/Q \in \mathbb{F}$ and for k a positive constant.

ContinuedFraction:ApproximationsInDefect

Let $\xi = [0; b_1, b_2, \ldots]$ be a continued fraction (either finite or infinte) which converges towards a number α and let A_n / B_n denote the nth convergent of ξ, $n = 1, 2, \ldots$. Then the odd convergents A_{2n-1}/B_{2n-1}, $n = 1, 2, \ldots$, which increase towards α are called approximations in defect.

ContinuedFraction:ApproximationsInExcess

Let $\xi = [0; b_1, b_2, \ldots]$ be a continued fraction (either finite or infinte) which converges towards a number α and let A_n / B_n denote the nth convergent of ξ, $n = 1, 2, \ldots$. Then the even convergents A_{2n}/B_{2n}, $n = 1, 2, \ldots$, which decrease towards α are called approximations in excess.

ContinuedFraction:AssociatedContinuedFraction

Given sequences of complex numbers α_n and β_n with $\alpha_n \neq 0$ the associated continued fraction is the generalized continued fraction

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{\begin{cases} \alpha_n & \text{for } n = 1 \\ -z^2 \alpha_n & \text{otherwise} \end{cases}}{1 + \beta_n z}.$$

ContinuedFraction:BaumSweetContinuedFraction

Let $s = \{s_n\}_{n=1}^{\infty}$ be a binary sequence whose nth term s_n is defined to be 0 if the binary expansion n contains (at least) one string of zeros having odd length and is defined to equal 1 otherwise. The sequence s is called the Baum-Sweet sequence and the regular continued fraction $\xi = [0; s_0, s_1, s_2, \ldots]$ is called the Baum-Sweet fraction associated to s. This construction can be also generalized by way of the transformation $0 \mapsto a$, $1 \mapsto b$ for distinct positive integers $a, b \in \mathbb{Z}^+$, whereby $s_k \in \{a, b\}$ for all $k = 0, 1, 2, \ldots$.

Though this fraction seems to be the focus of relatively little literature, it was defined by Baum and Sweet as part of their work on algebraic power series and has been linked to areas such as diophantine approximation theory. Moreover, one of the more well-known properties of the Baum-Sweet fraction ξ is that it is transcendental, a result which can be proved by advanced numerical methods found, e.g., in the work of Adamczewski.

ContinuedFraction:ByExcessContinuedFraction

A continued fraction is called a by-excess continued fraction if it has the form

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{-1}{b_k},$$

where $b_k \in \mathbb{Z}^+$ and N is possibly ∞.

For example,

$$\frac{1531}{1101} = 2 - \cfrac{1}{2 - \cfrac{1}{3 - \cfrac{1}{5 - \cfrac{1}{3 - \cfrac{1}{4 - \frac{1}{5}}}}}}.$$

is a by-excess continued fraction.

ContinuedFraction:CConvergent

For any irrational $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ with associated C-regular continued fraction ξ of the form

$$\xi = 2\, a_0 - 1 + \cfrac{\epsilon_1}{a_1 + \cfrac{1}{a_2 + \cfrac{\epsilon_3}{a_3 + \cfrac{1}{a_4 + \cdots}}}},$$

the ratios A_n / B_n for all natural numbers $n \in \mathbb{Z}^+$ are called the C-convergents of ξ.

ContinuedFraction:CDualConvergent

For any irrational $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ with associated C-dually regular continued fraction ξ of the form

$$\xi = 2\, b_0 - 1 + \cfrac{1}{b_1 + \cfrac{\epsilon_2}{b_2 + \cfrac{1}{b_3 + \cfrac{\epsilon_4}{b_4 + \cdots}}}},$$

the ratios A_n / B_n for all natural numbers $n \in \mathbb{Z}^+$ are called the C-dual convergents of ξ.

ContinuedFraction:CDuallyRegularFraction

Let ξ be a continued fraction of the form

$$\xi = 2\,b_0 - 1 + \cfrac{2}{2\,b_1 + \cfrac{2\,\epsilon_2}{2\,b_2 + \cfrac{2}{2\,b_3 + \cfrac{2\,\epsilon_4}{2\,b_4 + \cdots}}}}$$

where $b_0 \in \mathbb{Z}$, $b_n \in \mathbb{Z}^+$, and $\epsilon_{2\,n}$ satisfies

$$\epsilon_{2\,n} = \begin{cases} +1 & \text{for } U_{2\,n} = C \\ -1 & \text{for } U_{2\,n} = E_1 \end{cases}$$

for all $n \in \mathbb{Z}^+$. Here, the elements U_i come from the dually regular chain representation

$$V_1^{b_0-1}\,C\,V_1^{b_1-1}\,U_2\,V_1^{b_2-1}\,C\,V_1^{b_3-1}\,U_4\,V_1^{b_4-1}\,\cdots$$

of a related complex number ξ_0^* and the matrices V_1, C, and E_1 are defined to be

$$V_1 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & i - 1 \\ 1 - i & i \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1 - i & i \end{pmatrix}.$$

ContinuedFraction:CDuallyRegularPurelyPeriodicFraction

A C-dually regular continued fraction ξ of the form

$$\xi = 2\,b_0 - 1 + \cfrac{2}{2\,b_1 + \cfrac{2\,\epsilon_2}{2\,b_2 + \cfrac{2}{2\,b_3 + \cfrac{2\,\epsilon_4}{2\,b_4 + \cdots}}}}$$

is said to be purely periodic if both sequences $\{b_1, b_3, b_5, \ldots\}$ and $\{\epsilon_2, \epsilon_4, \epsilon_6, \ldots\}$ are both purely periodic.

ContinuedFraction:CFraction

A generalized continued fraction ξ_C is called a C-fraction if it has the form

$$\xi_C = b_0 + \cfrac{a_1 z^{\alpha_1}}{1 + \cfrac{a_2 z^{\alpha_2}}{1 + \cfrac{a_3 z^{\alpha_3}}{1 + \cdots}}},$$

where $b_0 \in \mathbb{C}$ is an arbitrary complex number and where a_n and α_n are sequences of nonzero complex numbers and of integers, respectively. This definition can be made more precise, however.

Let $P(z) = c_0 + c_1 z + c_2 z^2 + \cdots$, $c_0 \neq 0$, be a formal power series with coefficients $c_k \in \mathbb{C}$. The generalized continued fraction ξ_C of the form

$$\xi_C = c_0 + \cfrac{a_1 z^{\alpha_1}}{1 + \cfrac{a_2 z^{\alpha_2}}{1 + \cfrac{a_3 z^{\alpha_3}}{1 + \cdots}}}$$

is said to be the "corresponding continued fraction" to P (i.e., a C-fraction) provided its elements satisfy the "correspondence relations"

$$(c_n, c_{n-1}, c_{n-2}, \ldots) \begin{pmatrix} \delta_{p,0} \\ \delta_{p,1} \\ \delta_{p,2} \\ \vdots \end{pmatrix} =$$

$$\begin{cases} 0 & \text{for } \alpha_0 + \cdots + \alpha_p < n < \alpha_1 + \cdots + \alpha_{p+1} \\ (-1)^p a_1 a_2 \cdots a_{p+1} & \text{for } n = \alpha_1 + \cdots + \alpha_{p+1}, \end{cases}$$

where $\delta_{i,j}$ denotes Kronecker's delta.

In some ways, C-fractions appear to be the most far-reaching of the families of fractions generally defined, though as their definition suggests, they appear particularly often in literature on the theories of formal power series. They also appear quite frequently in the study of Padé approximants, so much so that subclasses of C-fractions (regular C-fractions, for example) are often classified and studied based on correspondences with Padé approximants.

ContinuedFraction:ChanExpansion

Let $0 \le \xi < 1$ be a real number. For any integer $m \ge 2$, Chan's continued fraction expansion is defined through

$$\xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{\delta_{j,1}\, m^{-a_1} + \left(1 - \delta_{j,1}\right)(m-1)\, m^{-a_j}}{1}.$$

The coefficients a_n can be calculated through

$$a_n = \eta_m\left(\tau_m^{n-1}(\xi)\right)$$

where

$$\eta_m(x) = \begin{cases} \left\lfloor -\log_m(x) \right\rfloor & \text{for } x \ne 0 \\ \infty & \text{for } x = 0 \end{cases}$$

and

$$\tau_m(x) = \begin{cases} \frac{1}{m-1}\left(\frac{1}{m^i\, x} - 1\right) & \text{when } \exists_{i, i \in \mathbb{Z}^+}\ \frac{1}{m^{i+1}} < x \le \frac{1}{m^i} \\ 0 & \text{for } x = 0. \end{cases}$$

ContinuedFraction:CommonNotations

Common notations for the generalized continued fraction

$$\xi = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}$$

include

$$\xi = b_0 + \frac{a_1}{b_1 +}\ \frac{a_2}{b_2 +}\ \frac{a_3}{b_3 +} \ldots$$

$$\xi = b_0 + \left.\frac{a_1}{b_1}\right| + \left.\frac{a_2}{b_2}\right| + \left.\frac{a_3}{b_3}\right| + \ldots \quad \text{(Pringsheim)}$$

$$\left(a_1, a_2, a_3, \ldots;\ b_0, b_1, b_2, b_3, \ldots\right) \quad \text{(Leighton and Wall)}$$

and

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k} \quad \text{(Gauss).}$$

In Gauss's notation, the uppercase K stands for "Kettenbruch," which is German for "continued fraction."

Common notations for the nth convergent of a continued fraction include p_n/q_n and A_n/B_n, the former being more prevalent in older papers and the latter being more common in the recent literature. Here, the notation A_n/B_n is used.

ContinuedFraction:ComplexContinuedFraction

A continued fraction ξ of the form

$$\xi = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}}$$

is said to be a complex continued fraction if for each $k = 1, 2, 3, \ldots, a_k, b_k \in \mathbb{C}$.

ContinuedFractionConditionForTrivialClassNumber

Let d be a squarefree integer, F be its quadratic field, and n be the class number set of F. Given

$d \bmod 4 = 2 \bigvee d \bmod 4 = 3$

then $n = 1$ if and only if d has the monadic expansion property.

ContinuedFraction:Continuant

The multivariate polynomials K_n (continuants or continuant polynomials) are defined through

$K_0() = 1$

$K_1(x_1) = x_1$

$K_n(x_1, \ldots, x_n) = K_{n-1}(x_1, \ldots, x_{n-1})\, x_n + K_{n-2}(x_1, \ldots, x_{n-2})$ for $n \geq 2$.

ContinuedFraction:Convergence

A continued fraction ξ of the form

$$\xi = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}}$$

with nth convergent ξ_n is said to converge to a value x if $\xi_n \to x$ as $n \to \infty$. In the case where $\xi_n \to \pm\infty$, ξ is said to be inessentially divergent; if $\lim_{n\to\infty} \xi_n$ fails to exist, ξ is said to be essentially divergent.

Note that $\xi \to x$ as $n \to \infty$ occurs precisely when $\xi_{2n} \to x$ and $\xi_{2n-1} \to x$ as $n \to \infty$. Also, while notationally similar to convergence of a real sequence, say, continued fraction convergence is considerably different. Unlike with convergent series, for example, omission of a finite number of initial terms of a continued fraction can completely change convergence-related behavior.

ContinuedFraction:Convergent

Given a continued fraction ξ of the form

$$\xi = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}},$$

its nth convergent ξ_n is the finite continued fraction obtained by truncating ξ at the nth level, i.e.,

$$\xi_n = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{\ddots + \frac{a_n}{b_n}}}}.$$

Writing $\xi_n = A_n / B_n$, it is easily verified that the partial numerators and denominators of ξ_n satisfy the recurrence relations $A_n = a_n A_{n-1} + b_n A_{n-2}$, $B_n = a_n B_{n-1} + b_n B_{n-2}$ for $n = 1, 2, \ldots$ provided one defines $A_{-1} = 1$, $A_0 = b_0$, $B_{-1} = 0$, and $B_0 = 1$.

ContinuedFraction:ConvergentDenominator

Given a continued fraction ξ of the form

$$\xi = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}},$$

its nth convergent denominator B_n is the expression in the denominator of the nth convergent $\xi_n = A_n / B_n$ where ξ_n is the finite continued subfraction of the form

$$\xi_n = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{\ddots + \frac{a_n}{b_n}}}}.$$

ContinuedFraction:ConvergentNumerator

Given a continued fraction ξ of the form

$$\xi = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \cdots}}},$$

its nth convergent numerator A_n is the expression in the numerator of the nth convergent $\xi_n = A_n / B_n$ where ξ_n is the finite continued subfraction of the form

$$\xi_n = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{\ddots + \frac{a_n}{b_n}}}}.$$

ContinuedFraction:ConvergentRecurrenceRelations

The nth convergent $\xi_n = A_n / B_n$ of a generalized continued fraction $\xi = b_0 + \mathrm{K}\,(a_m / b_m)$ consists of elements A_n, B_n which satisfy the recurrence relations $A_n = b_n\, A_{n-1} + a_n\, A_{n-2}$, $B_n = b_n\, B_{n-1} + a_n\, B_{n-2}$, $n = 1, 2, 3, \ldots$, subject to the initial conditions $A_{-1} = B_0 = 1$, $B_{-1} = 0$, $A_0 = b_0$. Modulo the initial conditions, this recurrence relation can be written in shorthand by way of matrix operations, namely

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = b_n \begin{pmatrix} A_n - 1 \\ B_{n-1} \end{pmatrix} + a_n \begin{pmatrix} A_{n-2} \\ B_{n-2} \end{pmatrix}$$

for $n = 1, 2, 3, \ldots$.

The above-mentioned identity is a specialized case of the more general theory of three-term recurrence relations. Indeed, a sequence $\{X_n\}_{n=-1}^{\infty}$ of complex numbers is a solution of the so-called three-term recurrence relation

$$X_n = b_n\, X_{n-1} + a_n\, X_{n-2}$$

provided that all consecutive triples of its elements are solutions where, here, $a_n, b_n \in \mathbb{C}$ for $n = 1, 2, 3, \ldots$ and $a_k \neq 0$ for all k. In addition to the identity given above, one can easily show that the sequences $\{A_n\}$, $\{B_n\}$ associated to ξ actually form a basis for the solution space L of the three-term recurrence relation. A considerable amount of information concerning the role of contin∴ ued fractions in three-term recurrence relations and minimal solutions thereto can be found in the works of Pincherle and Gautschi.

ContinuedFraction:CRegularFraction

Let ξ be a continued fraction of the form

$$\xi = 2\,a_0 - 1 + \cfrac{2\,\epsilon_1}{2\,a_1 + \cfrac{2}{2\,a_2 + \cfrac{2\,\epsilon_3}{2\,a_3 + \cfrac{2}{2\,a_4 + \cdots}}}}$$

where $a_0 \in \mathbb{Z}$, $a_n \in \mathbb{Z}^+$, and $\epsilon_{2\,n-1}$ satisfies

$$\epsilon_{2\,n-1} = \begin{cases} +1 & \text{for } U_{2\,n-1} = C \\ -1 & \text{for } U_{2\,n-1} = E_1 \end{cases}$$

for all $n \in \mathbb{Z}^+$. Here, the elements U_i come from the regular chain representation

$$V_1^{a_0-1}\,U_1\,V_1^{a_1-1}\,C\,V_1^{a_2-1}\,U_3\,V_1^{a_3-1}\,C\,V_1^{a_4-1}\,\cdots$$

of a related complex number ξ_0 and the matrices V_1, C, and E_1 are defined to be

$$V_1 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & i-1 \\ 1-i & i \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix}.$$

ContinuedFraction:CRegularPurelyPeriodicFraction

A C-regular continued fraction ξ of the form

$$\xi = 2\,a_0 - 1 + \cfrac{2\,\epsilon_1}{2\,a_1 + \cfrac{2}{2\,a_2 + \cfrac{2\,\epsilon_3}{2\,a_3 + \cfrac{2}{2\,a_4 + \cdots}}}}$$

is said to be purely periodic if both sequences $\{a_0, a_1, a_2, \ldots\}$ and $\{\epsilon_1, \epsilon_3, \epsilon_5, \ldots\}$ are both purely periodic.

ContinuedFraction:Divergence

Divergence of a continued fraction ξ of the form

$$\xi = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

with nth convergent ξ_n occurs when ξ_n fails to converge to a finite expression as $n \to \infty$.

Two distinct types of divergence are defined: In the case where $\xi_n \to \pm\infty$, ξ is said to be inessentially divergent while ξ is said to diverge essentially provided that $\lim_{n\to\infty} \xi_n$ fails to exist. Essential divergence can be examined by considering the even and odd convergents ξ_{2n} and ξ_{2n-1} of ξ, respectively, and in particular, ξ will essentially divergent provided that either of $\lim_{n\to\infty} \xi_{2n}$, $\lim_{n\to\infty} \xi_{2n-1}$ fails to exist or in the case that both limits exist but are unequal.

ContinuedFraction:EllipticContinuedFraction

A p-periodic continued fraction $\xi = \mathrm{K}(a_n / b_n)$ is said to be elliptic if S_p is elliptic, i.e., if $|\mathcal{R}| = |\mathcal{R}(\xi)| = 1$, $\mathcal{R} \neq 1$. Here, S_n is the Möbius transformation defined for all $w \in \mathbb{C}$ by the approximant function

$$S_n(w) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}$$

and $\mathcal{R}$ is the ratio

$$\mathcal{R} = \begin{cases} \frac{1-u}{1+u} & \text{for } c \neq 0,\ a + d \neq 0 \\ -1 & \text{for } c \neq 0,\ a + d = 0 \end{cases}$$

associated to $S_n = (a\,w + b)/(c\,w + d)$ where $u = \sqrt{1 - 4\,\Delta \big/ (a + d)^2}$, $\Delta = a\,d - b\,c \neq 0$.

ContinuedFraction:EulerFraction

Given a sequence of complex numbers α_n with $\alpha_n \neq 0$, the Euler fraction is the generalized continued fraction

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{-\alpha_n\, z}{1 + \begin{cases} 0 & \text{for } n = 1 \\ z\,\alpha_n & \text{otherwise} \end{cases}}.$$

ContinuedFraction:EvenPart

Let $x = [b_0; b_1, b_2, \ldots]$ be an arbitrary regular continued fraction whose kth approximant is denoted $x_k = A_k(x)/B_k(x)$. Then the even part of x is the sequence $\{x_2, x_4, \ldots\}$ of the even approximants of x.

ContinuedFraction:Expansion

Given a constant c, a regular continued fraction expansion is an expression of the form

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k}$$

with partial numerators a_k and partial denominators b_k taken from some domain, usually positive integers, such that $\xi = c$.

ContinuedFractionExpansionHurwitzApproximationQuality

Let z be a complex number with $-1/2 < \mathrm{Re}(z) < 1/2$ and $-1/2 < \mathrm{Im}(z) < 1/2$ with Hurwitz continued fraction expansion

$$z = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}$$

with N possibly infinite and A_n/B_n the sequence of convergents. Suppose B is a Gaussian integer with $|B_{n-1}| < B < |B_{n+1}|$ and A is a Gaussian integer with $A/B \neq A_n/B_n$. Then

$$\left|z - \frac{A}{B}\right| \geq \frac{1}{5}\left|z - \frac{A_n}{B_n}\right|\left|\frac{B_n}{B}\right|$$

for all n.

ContinuedFractionExpansionHurwitzBoundedPartialDenominators

Let z be a complex number with Hurwitz continued fraction expansion

$$z = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then for every even integer d there exist nonreal algebraic numbers of degree d over $\mathbb{Q}$ such that the Hurwitz expansion has bounded partial denominators b_k.

ContinuedFractionExpansionHurwitzConvergentDenominatorGrowth

Let z be a complex number with $-1/2 < \mathrm{Re}(z) < 1/2$ and $-1/2 < \mathrm{Im}(z) < 1/2$ with Hurwitz continued fraction expansion

$$z = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}$$

with N possibly infinite. Then the denominators of the convergents A_n / B_n satisfy

$$\frac{|B_{n+2}|}{|B_n|} \geq \frac{3}{2}$$

for all positive integer $n \leq N$.

ContinuedFractionExpansionHurwitzQuadratic

Let z be a complex number that is the root of a quadratic equation with Gaussian integer coefficients with Hurwitz continued fraction expansion

$$z = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

The b_j are defined through

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right.$$

$$b_0 = \lfloor x$$

$$b_j = \left\lfloor \frac{1}{\tau^j(x)} \right. .$$

Then only finitely many different $\tau^j(x)$ exist for $x = z$.

ContinuedFractionExpansionsOfRational

Let α be a rational number where $0 < \alpha < 1$; then it has exactly two regular continued fractions that are finite:

$$\xi_1 = \underset{n=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_{1,n}}$$

and

$$\xi_2 = \underset{n=1}{\overset{N+1}{\mathrm{K}}} \frac{1}{b_{2,n}}$$

where

$$\left(b_{1,N} > 0 \land \forall_{n>N}\, b_{1,n} = 0 \land \forall_{n>1+N}\, b_{2,n} = 0 \land \forall_{n<N}\, b_{1,n} = b_{2,n} \land -1 + b_{1,n} = b_{2,n} \land b_{2,N+1} = 1\right).$$

ContinuedFraction:FamilyTypes

A continued fraction ξ_C is called a C-fraction if it has the form

$$\xi_C = b_0 + \frac{a_1 z^{\alpha_1}}{1 + \frac{a_2 z^{\alpha_2}}{1 + \frac{a_3 z^{\alpha_3}}{1 + \cdots}}},$$

where $b_0 \in \mathbb{C}$ is an arbitrary complex number and where a_n and α_n are sequences of nonzero complex numbers and of integers, respectively. The "C" stands for "corresponding type," as fractions of this form correspond to formal power series $P(z) = c_0 + c_1 z + c_2 z^2 + \cdots$, $c_0 \neq 0$, $c_k \in \mathbb{C}$.

Given complex sequences $a_1, a_2, \ldots \neq 0$ and $b_1, b_2, \ldots$, the continued fraction ξ_J is said to be a J-fraction or Jacobi-fraction provided it has the form

$$\xi_J = \frac{a_1}{z + b_1 - \frac{a_2}{z + b_2 - \frac{a_3}{z + b_3 - \cdots}}}.$$

The continued fraction ξ_M is said to be an M-fraction provided that, for sequences of complex numbers F_n, $G_n \in \mathbb{C}$,

$$\xi_M = \frac{F_1}{1 + G_1 z + \frac{z F_2}{1 + G_2 z + \frac{z F_3}{1 + G_3 z + \cdots}}}.$$

Fractions ξ_{Th} of the form

$$\xi_{\mathrm{Th}} = \frac{c_1 z}{e_1 + d_1 z + \frac{c_2 z}{e_2 + d_2 z + \frac{c_3 z}{e_3 + d_3 z + \cdots}}}$$

are said to be a generalized Thron fractions when $d_n \in \mathbb{C}$, c_n, $e_n \in \mathbb{C} \setminus \{0\}$ for $n = 1, 2, 3, \ldots$. They can be further classified by examining c_n, d_n, e_n:

- Fractions with $e_n = 1$, $c_n = F_n$, and $d_n = G_n$ are called Thron fractions or generalized T-fractions.
- Thron fractions with $F_n = 1$ for all n are called T-fractions.

• Thron fractions for which $F_m, G_m > 0$ for all m are positive T-fractions.

• Thron fractions for which $F_m, G_m \in \mathbb{R} \setminus \{0\}$ satisfy the conditions $F_{2m-1} F_{2m} > 0$, $F_{2m-1}/G_{2m-1} > 0$ are called alternating positive term fractions (APT) fractions.

The continued fractions ξ_S of the form

$$\xi_S = \frac{a_1 z}{1 + \frac{a_2 z}{1 + \frac{a_3 z}{1 + \cdots}}}$$

are called Stieltjes-fractions or S-fractions provided $a_n \in \mathbb{R}^+$. Any continued fraction f which satisfies $B(f(a(z))) = g(z)$ is called a modified S-fraction for g an S-fraction, $a, B : \Omega \subset \mathbb{C} \to \mathbb{C}$ meromorphic functions.

A continued fraction ξ_P is said to be a P-fraction if

$$\xi_P = b_0 \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots}}}$$

where for each $n = 0, 1, 2, \ldots, b_n = b_n(1/z)$ is a polynomial in $1/z$.

Given a function f, the Thiele-fraction is an interpolating fraction ξ_{appr} of the form

$$\xi_{\text{appr}} = b_0 + \frac{z - z_0}{b_1 + \frac{z - z_1}{b_2 + \frac{z - z_2}{b_3 + \cdots}}}$$

where here, the elements z_n are distinct points at which values of f are known and the elements b_n are formed from the inverse differences of f: $b_0 = \varphi_0[z_0]$ and $b_k = \varphi_k[z_0, \ldots, z_k]$ for $k = 1, 2, \ldots$.

A fraction of the form

$$\xi_{\lambda_q} = b_0 \lambda_q + \frac{\epsilon_1}{\lambda_q b_1 + \frac{\epsilon_2}{\lambda_q b_2 + \frac{\epsilon_3}{\lambda_q b_3 + \cdots}}}$$

is said to be a λ_q-fraction provided that for $q \geq 3$ odd, $\lambda_q = 2\cos(\pi/q)$, $b_0 \in \mathbb{Z}$, $b_n \in \mathbb{Z}^+$ for $n = 1, 2, \ldots$, and $\epsilon_n \in \{\pm 1\}$ for $n \geq 0$. When $q = 5$, $\lambda_q = \phi$, and the resulting fraction $\xi_{\lambda_5} = \xi_\tau$ is said to be a τ-fraction.

ContinuedFraction:FiniteDerivative

Given a finite generalized continued fraction $\xi_{n,N} = \xi_{n,N}(z)$ of the form

$$\xi_{n,N} = \cfrac{a_n}{b_n + \cfrac{a_{n+1}}{b_{n+1} + \cfrac{a_{n+2}}{\ddots + \cfrac{a_N}{b_N}}}},$$

where $a_k = a_k(z)$ and $b_k = b_k(z)$ are complex-valued analytic functions for all $k = n, n+1, n+2, \ldots, N$, the derivative of $\xi_{n,N}$ with respect to z is given by

$$\frac{d}{dz}\left(\underset{k=n}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}\right) = \sum_{j=n}^{N} (-1)^{j-n+1} \left(\prod_{k=n}^{j} \frac{1}{a_k} \left(\underset{\ell=k}{\overset{N}{\mathrm{K}}} \frac{a_\ell}{b_\ell}\right)^2\right) \left(\left(\underset{\ell=j}{\overset{N}{\mathrm{K}}} \frac{a_\ell}{b_\ell}\right)^{-1} \frac{d a_j}{dz} - \frac{d b_j}{dz}\right).$$

Moreover, $d\xi_{n,N} / dz$ is an analytic function for all z for which $(z, 0) \in G$, where

$$G \subset \Psi \bigcap \Omega \times B(0, R_G) \subset (\mathbb{C} \bigcup \{\infty\})^2$$

is the domain of analyticity of the sequence $\{g_k(z, \zeta)\}$ defined by

$$g_k(z, \zeta) = \left(g_{k,1}(z), g_{k,2}(z, \zeta)\right) = \left(z, \frac{a_k(z)}{b_k(z) + \zeta}\right),$$

where Ψ, respectively Ω, is the domain of analyticity for the sequence $\{a_k(z)\}$, respectively $\{b_k(z)\}$, and where $R_G < \infty$ is some positive radius.

ContinuedFraction:FinitePartialDerivative

Given sequences $\{a_k\}_{k=1}^{\infty} = \{a_k(z)\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty} = \{b_k(z)\}_{k=1}^{\infty}$ of complex-valued functions analytic on domains Ψ and Ω, respectively, for which $a_k \neq 0$ for $k < N$ for some N and in which all a_k and $b_{k\neq\ell}$ are constant, applying $dz = (\partial b_\ell/\partial z)^{-1}\, d b_\ell$ to the derivative formula

$$\frac{d}{dz}\left(\underset{k=n}{\overset{\infty}{\mathrm{K}}}\frac{a_k}{b_k}\right) = \sum_{j=n}^{N}(-1)^{j-n+1}\left(\prod_{k=n}^{j}\frac{1}{a_k}\left(\underset{\ell=k}{\overset{N}{\mathrm{K}}}\frac{a_\ell}{b_\ell}\right)^2\right)\left(\left(\underset{\ell=j}{\overset{N}{\mathrm{K}}}\frac{a_\ell}{b_\ell}\right)^{-1}\frac{d a_j}{dz} - \frac{d b_j}{dz}\right)$$

yields

$$\frac{\partial}{\partial b_\ell}\left(\underset{k=1}{\overset{N}{\mathrm{K}}}\frac{a_k}{b_k}\right) = (-1)^\ell \prod_{k=1}^{\ell}\frac{1}{a_k}\left(\underset{j=k}{\overset{N}{\mathrm{K}}}\frac{a_j}{b_j}\right)^2$$

in the event that neither b_ℓ nor $d b_\ell/dz$ vanishes. The so-called determinant formula along with the three-term recurrence relation

$B_m = b_m\, B_{m-1} + a_m\, B_{m-2}$,

$B_{-1} = 0$, $B_0 = 1$, satisfied by the finite convergents of $\underset{k=1}{\overset{\infty}{\mathrm{K}}}(a_k/b_k)$ allows this partial derivative expression to be rewritten as

$$\frac{\partial}{\partial b_\ell}\left(\underset{k=1}{\overset{N}{\mathrm{K}}}\frac{a_k}{b_k}\right) = -\frac{\left(\underset{k=1}{\overset{\ell-1}{\mathrm{K}}}\frac{a_k}{b_k} - \underset{k=1}{\overset{N}{\mathrm{K}}}\frac{a_k}{b_k}\right)^2 B_{\ell-1}}{\left(\underset{k=1}{\overset{\ell}{\mathrm{K}}}\frac{a_k}{b_k} - \underset{k=1}{\overset{\ell-1}{\mathrm{K}}}\frac{a_k}{b_k}\right) B_\ell}.$$

ContinuedFraction:FormalDenominator

Let ξ be a regular continued fraction of the form $\xi = [a_1, a_2, \ldots, a_n]$ whose successive quotients a_k are taken from either $\mathbb{R}$ or $\mathbb{C}$ for $k = 1, 2, \ldots$. The nth formal denominator of ξ is then the element B_n in the identity

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} A_n \\ B_n \end{pmatrix}.$$

ContinuedFraction:FormalNumerator

Let ξ be a regular continued fraction of the form $\xi = [a_1, a_2, \ldots, a_n]$ whose successive quotients a_k are taken from either $\mathbb{R}$ or $\mathbb{C}$ for $k = 1, 2, \ldots$. The nth formal numerator of ξ is then the element A_n in the identity

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} A_n \\ B_n \end{pmatrix}.$$

ContinuedFraction:GeneralConvergence

A generalized continued fraction $\underset{k=1}{\overset{\infty}{\mathrm{K}}} a_k / b_k$ converges generally to a value $f \in \hat{\mathbb{C}}$ if there exist two sequences $\{v_n\}_{n=1}^{\infty}$ and $\{w\}_{n=1}^{\infty}$ of extended complex numbers such that

$$\lim_{n\to\infty} S_n(v_n) = \lim_{n\to\infty} S_n(w_n) = f$$

and

$$\liminf_{n\to\infty} S(w_m,\ v_n) > 0,$$

where $S_m(w)$ is the nth approximant and $S(w_m,\ v_n)$ denotes the chordal metric on the extended complex plane $\hat{\mathbb{C}}$.

ContinuedFraction:GeneralThronFraction

A generalized continued fraction ξ of the form

$$\xi = \frac{c_1\, z}{e_1 + d_1\, z + \frac{c_2\, z}{e_2 + d_2\, z + \frac{c_3\, z}{e_3 + d_3\, z + \cdots}}}$$

is said to be a generalized Thron fraction provided that $d_n \in \mathbb{C}$ and $c_n,\, e_n \in \mathbb{C} \setminus \{0\}$ for $n = 1,\, 2,\, 3,\, \ldots$. Note that the "standard" Thron fraction is a specific case where $e_n = 1$, $c_n = F_n$, and $d_n = G_n$ for all n; similarly, the T-fraction results from further assuming that $c_n = F_n = 1$ for all n, and it follows that other subclasses of "standard" Thron fractions result from specifying certain restrictions to the elements of general Thron fractions ξ.

ContinuedFraction:GrommerFraction

A continued fraction ξ of the form

$$\xi = \frac{c_0}{z - \alpha_0 - \frac{\beta_0}{z - \alpha_1 - \frac{\beta_1}{z - \alpha_2 - \cdots - \frac{\beta_{r-2}}{z - \alpha_{r-1} - \cdots}}}}$$

is called a Grommer fraction if $\beta_k > 0$ for $k = 0,\, 1,\, 2,\, \ldots$.

ContinuedFraction:HalfRegularContinuedFraction

Given sequences of integers a_n, b_n with for $n > 0$, $b_n \geq 2$, $|a_n| = 1$ and $b_n + a_{n+1} \geq 2$, the half-regular fraction is the generalized continued fraction

$$b_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n}.$$

ContinuedFraction:HurwitzExpansion

The Hurwitz expansion of a complex number $z = a + b\,i \in \mathbb{C}$ is the complex continued fraction ξ of the form

$$\xi = b_0 + \underset{m=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_m},$$

$b_k \in \mathbb{C}$ for all k, $1 \leq N \leq \infty$, whose successive elements b_n are computed by way of the Hurwitz fraction algorithm. Explicitly, the Hurtwitz expansion ξ associated to z is computed recursively in terms of its nth partial denominators b_n by way of the recursion $b_0 = \lfloor z$ and

$$b_n = \left\lfloor \frac{1}{\tau^n(z)} \right.,$$

$n = 1, 2, 3, \ldots$, where $\lfloor z$ denotes the nearest Gaussian integer to z, $\tau(z)$ is the transformation $\tau(z) = 1/z - \lfloor 1/z$, and where $\tau^n(z)$ denotes the n-fold composition of τ with itself. The Hurwitz expansion is a popular alternative to the oft-studied Schmidt complex fraction expansion and tends to be preferred for its intuitiveness its computational simplicity.

ContinuedFraction:IdentityTypeContinuedFraction

A p-periodic continued fraction $\xi = \mathrm{K}(a_n/b_n)$ is said to be of identity type if S_p is the identity transformation, i.e., if $S_p(w) = \mathrm{Id}(w) = w$ for all $w \in \mathbb{C}$. Here, S_n is the Möbius transformation defined for all $w \in \mathbb{C}$ by the approximant function

$$S_n(w) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}.$$

ContinuedFraction:InfiniteDerivative

Given sequences $\{a_k\}_{k=1}^{\infty} = \{a_k(z)\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty} = \{b_k(z)\}_{k=1}^{\infty}$ of complex-valued functions analytic on domains Ψ and Ω, respectively, for which $a_k \neq 0$ for $k < N$ for some N and which are constants except for subsequences $\{a_\ell(z)\}_{\ell \in I}$, $\{b_\ell(z)\}_{\ell \in J}$, $I,\ J \in \{1,\ 2,\ \ldots\ ,\ N\}$, the $N+1$ tail $\underset{k=N+1}{\overset{\infty}{\mathrm{K}}} (a_k / b_k)$ is defined and converges to a value ζ, from which it follows that the infinite continued fraction ξ given by

$$\xi = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{\ddots}}}$$

has a derivative of the form

$$\frac{d}{dz}\left(\underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k}\right) = \sum_{j=1}^{\infty} (-1)^j \left(\prod_{k=1}^{j} \frac{1}{a_k} \left(\underset{\ell=k}{\overset{\infty}{\mathrm{K}}} \frac{a_\ell}{b_\ell}\right)^2\right)\left(\left(\underset{\ell=j}{\overset{\infty}{\mathrm{K}}} \frac{a_\ell}{b_\ell}\right)^{-1} \frac{d a_j}{dz} - \frac{d b_j}{dz}\right).$$

Moreover, the derivative $d\xi / dz$ is an analytic function for all z for which $(z,\ 0) \in G$ where here,

$G \subset \Psi \bigcap \Omega \times B(0,\ R_G) \subset (\mathbb{C} \bigcup \{\infty\})^2$

is the domain of analyticity of the sequence $\{g_k(z,\ \zeta)\}$ defined by

$$g_k(z,\ \zeta) = \left(g_{k,1}(z),\ g_{k,2}(z,\ \zeta)\right) = \left(z,\ \frac{a_k(z)}{b_k(z) + \zeta}\right),$$

where Ψ, respectively Ω, is the domain of analyticity for the sequence $\{a_k(z)\}$, respectively $\{b_k(z)\}$, and where $R_G < \infty$ is some positive radius.

ContinuedFraction:InfiniteFraction

An infinite continued fraction is a triple $[\{a_n\}_{n=1}^{\infty},\ \{b_n\}_{n=1}^{\infty},\ \{f_n\}_{n=1}^{\infty}]$ of sequences, where the a_k, b_k are given complex numbers with $a_n \neq 0$ for all n, and f_n is an element of the extended complex plane defined as follows.

Let s_n be the linear fractional transformation

$$s_n(z) = \frac{a_n}{b_n + z}$$

for $n \in \mathbb{Z}^+$, $S_n(z)$ the approximant function

$S_1(z) = s_1(z)$

$S_n(z) = S_{n-1}(s_n(z))$,

and

$f_n = S_n(0)$.

ContinuedFraction:InfinitePartialDerivative

Given sequences $\{a_k\}_{k=1}^{\infty} = \{a_k(z)\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty} = \{b_k(z)\}_{k=1}^{\infty}$ of complex-valued functions analytic on domains Ψ and Ω, respectively, for which $a_k \neq 0$ for $k < N$ for some N and in which all b_k and $a_{k \neq \ell}$ are constant, applying $dz = (\partial a_\ell / \partial z)^{-1} \, d a_\ell$ to the derivative formula

$$\frac{d}{dz}\left(\underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k}\right) = \sum_{j=1}^{\infty} (-1)^j \left(\prod_{k=1}^{j} \frac{1}{a_k} \left(\underset{\ell=k}{\overset{\infty}{\mathrm{K}}} \frac{a_\ell}{b_\ell}\right)^2\right) \left(\left(\underset{\ell=j}{\overset{\infty}{\mathrm{K}}} \frac{a_\ell}{b_\ell}\right)^{-1} \frac{d a_j}{dz} - \frac{d b_j}{dz}\right)$$

yields

$$\frac{\partial}{\partial a_\ell}\left(\underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k}\right) = \frac{1}{a_\ell}\left(\underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{a_j}{b_j}\right) \prod_{k=2}^{\ell} \frac{-\underset{j=k}{\overset{\infty}{\mathrm{K}}} \frac{a_j}{b_j}}{b_{k-1} + \underset{j=k}{\overset{\infty}{\mathrm{K}}} \frac{a_j}{b_j}}$$

in the event that neither a_ℓ nor $d a_\ell / dz$ vanishes.

ContinuedFraction:IntegerContinuedFraction

An integer continued fraction (or ICF) is a continued fraction ξ of the form

$$\xi = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

where $b_k \in \mathbb{Z}$ for each $k = 1, 2, \ldots$.

ContinuedFraction:IntegerPart

Given a generalized continued fraction

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k},$$

b_0 is known as the integer part.

ContinuedFraction:IntermediateConvergent

Let ξ be a regular continued fraction of the form

$$\xi = b_0 + \cfrac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots}}}$$

with nth convergent $\xi_n = A_n / B_n$, $n = 0, 1, 2, \ldots$, where $A_{-2} = 0$, $A_{-1} = 1$, $B_{-2} = 1$, and $B_{-1} = 0$ by definition. The intermediate convergents of ξ are a collection of expressions of the form

$$\xi_n^{(k)} = \frac{A_n^{(k)}}{B_n^{(k)}} = \frac{A_{n-2} + k\, A_{n-1}}{B_{n-2} + k\, B_{n-1}},$$

$k = 1, 2, \ldots , b_n - 1$, which lie between ξ_{n-2} and ξ_n for $n = 0, 1, 2, \ldots$. One can easily show that the collection $\left\{\xi_n^{(k)}\right\}$ is strictly increasing with respect to k.

ContinuedFraction:JFraction

Given complex sequences $a_1, a_2, \ldots \neq 0$ and $b_1, b_2, \ldots$, the generalized continued fraction ξ_J is said to be a J-fraction or Jacobi-fraction provided it has the form

$$\xi_J = \cfrac{a_1}{z + b_1 - \frac{a_2}{z + b_2 - \frac{a_3}{z + b_3 - \cdots}}}.$$

As it turns out, J-fractions are commonly-used tools in the theory of formal power series and are related to so-called C-fractions in very specific ways pertaining thereto. In fact, one well-known result shows that under certain conditions, a formal power series $F(z)$ has a C-fraction expansion if and only if it has a J-fraction expansion. J-fractions are also particularly relevant to the theory of moment problems, as well as in the study of orthogonality among families of polynomials.

ContinuedFraction:JFractionEquivalentPowerSeries

Let ξ be a J-fraction of the form

$$\xi = \cfrac{a_0}{b_1 + z - \frac{a_1}{b_2 + z - \frac{a_2}{b_3 + z - \cdots}}}$$

and let $A_k(z)$, respectively $B_k(z)$, denote the kth partial numerator, respectively denominator, of ξ so that the ratio $A_k(z) / B_k(z)$ denotes the kth approximant of ξ. The equivalent power series of the J-fraction ξ is the uniquely determined power series $P(1/z)$ whose expansion in descending powers of z agrees with the descending powers of z in $A_k(z) / B_k(z)$ for the first $2\,k$ terms, $k = 1, 2, 3, \ldots$.

ContinuedFraction:KPeriodicFraction

A general continued fraction $\xi = b_0 + \mathrm{K}\,(a_m / b_m)$ is said to be k-periodic for some fixed positive integer k if the sequences $\{a_m\}$ and $\{b_m\}$ are k-periodic after the first N elements, i.e., if $a_{N+k\,p+q} = a_{N+q}$ and $b_{N+k\,p+q} = b_{N+q}$ where $N \in \mathbb{Z}^+$ is fixed, $p \geq 1$, and $q \in \{1, 2, 3\}$. Explicitly, then, a three-periodic fraction ξ has the form

$$\xi = b_0 + \frac{a_1}{b_1 +}\;\frac{a_2}{b_2 +}\cdots\frac{a_N}{b_N +}\;\frac{a_{N+1}}{b_{N+1} +}\;\frac{a_{N+2}}{b_{N+2} +}\cdots\frac{a_{N+k}}{b_{N+k} +}\;\frac{a_{N+1}}{b_{N+1} +}\;\frac{a_{N+2}}{b_{N+2} +}\cdots\frac{a_{N+k}}{b_{N+k} +}\cdots$$

for some fixed natural number N.

k-periodicity plays a significant role, e.g., in studying continued fraction convergence, in particular the study of convergence by way of tail sequence analysis. Such ideas are explored in greater detail in the works of Lorentzen and Waadeland.

ContinuedFraction:LambdaSubQFraction

Let $\lambda_q = 2\cos(\pi/q)$ where $q \geq 3$ is an arbitrary odd integer. Given $b_0 \in \mathbb{Z}$, $b_n \in \mathbb{Z}^+$ for $n = 1, 2, \ldots$, and $\epsilon_n \in \{\pm 1\}$ for $n \geq 0$, one can define a generalized continued fraction ξ_{λ_q} called the a λ_q-fraction which has the form

$$\xi_{\lambda_1} = b_0\,\lambda_q + \cfrac{\epsilon_1}{\lambda_q\,b_1 + \cfrac{\epsilon_2}{\lambda_q\,b_2 + \cfrac{\epsilon_3}{\lambda_q\,b_3 + \cdots}}}.$$

By definition, λ_q-fractions are obvious generalizations of the τ-fraction (namely, the τ-fraction is merely the λ_5-fraction since the golden ratio $\phi = \lambda_5$); as a result, fractions of this form are useful in many of the same ways as the τ-fractions and tend to come about by way of studying algebraic number fields generated by elements of the form λ_q.

ContinuedFraction:Limit

Let ξ be a generalized continued fraction of the form

$$\xi = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

whose elements a_i and b_i are positive integers and let $\xi_n = A_n / B_n$ denote the nth convergent of ξ, i.e.

$$\xi_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n}}}}.$$

If the sequence $\xi_n = A_n / B_n$ converges to a real number α as $n \to \infty$, then ξ is said to represent α and α is said to be the limit of ξ.

ContinuedFraction:LimitPeriodicFraction

A limit periodic continued fraction is a continued fraction $\xi = \mathrm{K}(b_n / 1) = [0; b_1, b_2, \ldots]$ such that, for some complex number b, $\lim_{n \to \infty} b_n = b \neq \infty$.

ContinuedFraction:LoxodromicContinuedFraction

A p-periodic continued fraction $\xi = \mathrm{K}(a_n / b_n)$ is said to be loxodromic if S_p is loxodromic, i.e., if $|\mathcal{R}| = |\mathcal{R}(\xi)| < 1$. Here, S_n is the M ̈obius transformation defined for all $w \in \mathbb{C}$ by the approximant function

$$S_n(w) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}$$

and $\mathcal{R}$ is the ratio

$$\mathcal{R} = \begin{cases} \frac{1-u}{1+u} & \text{for } c \neq 0,\ a + d \neq 0 \\ -1 & \text{for } c \neq 0,\ a + d = 0 \end{cases}$$

associated to $S_n = (a\, w + b) / (c\, w + d)$ where $u = \sqrt{1 - 4\,\Delta / (a + d)^2}$, $\Delta = a\, d - b\, c \neq 0$.

ContinuedFraction:MFraction

The generalized continued fraction ξ_M is said to be an M-fraction provided that, for complex sequences $F_n,\ G_n \in \mathbb{C}$,

$$\xi_M = \cfrac{F_1}{1 + G_1\, z + \cfrac{z\, F_2}{1+G_2\, z+\cfrac{z\, F_3}{1+G_3\, z+\cdots}}}.$$

Defined similarly to the J-fraction, M-fractions correspond conceptually to the expansion of formal power series $F(z)$ at two points whereas the J- and C-fractions consist of expansions about a single point. First considered in the seminal paper by namesakes Murray and McCabe, M-fractions have proven especially useful in the approximation by rational functions of several large classes of functions.

ContinuedFraction:ModifiedSFraction

Given an S-fraction g along with meromorphic functions $a,\ B : \Omega \subset \mathbb{C} \to \mathbb{C}$, any (meromorphic) continued fraction f which satisfies $B\,(f(a(z))) = g\,(z)$ is called a modified S-fraction. Defined to extend the applicability of "standard" S-fractions, modified S-fractions maintain many of the same useful analysis-theoretic properties thereof while providing a wider range of generalized solutions to various types of problems including moment problems and problems pertaining to functions of Hankel, Bessel, etc.

ContinuedFraction:NearestIntegerContinuedFraction

For a real number $\alpha \in \mathbb{R}$, the nearest integer continued fraction (NICF) associated to α is the regular continued fraction ξ of the form

$$\xi = b_0 + \underset{m=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_m}$$

where, here, successive elements b_k, $k = 1, 2, \ldots$, are integers found using the NICF expansion algorithm. Explicitly, the NICF ξ associated to α is computed recursively in terms of its nth convergents ξ_n by way of the identity

$$\xi_n = b_n + \frac{\epsilon_{n+1}}{\zeta_{n+1}},$$

where $b_n \in \mathbb{Z}$ is the nearest integer to ζ_n, $\epsilon_{n+1} \in \{\pm 1\}$, $|\zeta_n - b_n| < 1/2$, and $\operatorname{sgn}(\epsilon_{n+1}) = \operatorname{sgn}(\zeta_n - b_n)$.

Given a real number α with known continued fraction expansion $\xi = [b_0, b_1, \ldots, b_n, \beta_{n+1}]$, $b_k \in \mathbb{Z}$ for $k = 1, 2, \ldots, n$, $\beta_{n+1} \in \mathbb{R}$, Hurwitz discovered a result for determining whether ξ is the NICF expansion of α. In particular, ξ is the NICF expansion of α precisely when:

1. $|b_k| \geq 2$ for $k = 1, 2, \ldots, n$
2. b_{i+1} is negative when $b_i = 2$ and is positive when $b_i = -2$
3. $\beta_{n+1} \geq 2$ or $\beta_{n+1} < -2$ and $|b_n - 1/\beta_{n+1}| > 2$.

ContinuedFraction:ParabolicContinuedFraction

A p-periodic continued fraction $\xi = \mathrm{K}(a_n/b_n)$ is said to be parabolic if S_p is parabolic, i.e., if $\mathcal{R} = \mathcal{R}(\xi) = 1$, $S_p \neq \mathrm{Id}$. Here, S_n is the M bius transformation defined for all $w \in \mathbb{C}$ by the approximant fucntion

$$S_n(w) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}$$

and $\mathcal{R}$ is the ratio

$$\mathcal{R} = \begin{cases} \frac{1-u}{1+u} & \text{for } c \neq 0,\ a + d \neq 0 \\ -1 & \text{for } c \neq 0,\ a + d = 0 \end{cases}$$

associated to $S_n = (a\,w + b)/(c\,w + d)$ where $u = \sqrt{1 - 4\,\Delta\big/(a + d)^2}$, $\Delta = a\,d - b\,c \neq 0$.

ContinuedFraction:PartialDenominator

The partial denominators of a generalized continued fraction ξ of the form

$$\xi = b_0 + \underset{m=1}{\overset{N}{\mathrm{K}}} \frac{a_m}{b_m}$$

are the elements b_k, $k = 0, 1, 2, \ldots$.

ContinuedFraction:PartialNumerator

The partial numerators of a generalized continued fraction ξ of the form

$$\xi = b_0 + \underset{m=1}{\overset{N}{\mathrm{K}}} \frac{a_m}{b_m}$$

are the elements a_k, $k = 1, 2, 3, \ldots$.

ContinuedFraction:PerronCaratheodoryContinuedFraction

Given sequences of complex numbers α_n, β_n with $\alpha_n \neq 0$ and $\alpha_{2n+1} = 1 - \beta_{2n}\,\beta_{2n+1}$, the Perron-Carathéodory continued fraction is the generalized continued fraction

$$\beta_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{\begin{cases} \alpha_1 & \text{for } n = 1 \\ \alpha_n\, z & \text{for } n > 1 \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases}}{\begin{cases} \beta_n\, z & \text{for } n \text{ even} \\ \beta_n & \text{for } n \text{ odd} \end{cases}}.$$

ContinuedFraction:PFraction

A generalized continued fraction ξ_P is said to be a P-fraction if

$$\xi_P = b_0 \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots}}},$$

where for each $n = 0, 1, 2, \ldots$, $b_n = b_n(1/z)$ is a polynomial in $1/z$. Symbolically, then, one can think of the elements b_n of ξ_P to be of the form

$$b_n = \sum_{m=-N_n}^{0} a_{-m}^{(n)} z^m, \; n = 0, 1, 2, \ldots ,$$

where $N_n \geq 1$ and $a_{N_n}^{(n)} \neq 0$ for $n = 1, 2, 3, \ldots$. Continued fractions of this type emerged as part of the work of Magnus while attempting to create a theory of fractional expansions of meromorphic functions analogous to the theory of continued fraction expansions of real numbers. The name P-fraction refers to the fact that, for all n, the continued fraction $[b_n; b_{n+1}, b_{n+2}, \ldots]$ is defined to be the so-called principal part expansion for the Laurent power series $L_n(z)$ where

$$L_n(z) = \sum_{m=-N_n}^{\infty} a_{-m}^{(n)} z^m, \; n = 0, 1, 2, \ldots .$$

P-fractions are also related to the study of Padé approximants.

ContinuedFraction:PippengerFraction

A Pippenger continued fraction is a continued fraction of the form

$$\xi = 1 + \frac{1}{-1 + t_1\left(1 + \frac{1}{-1 + t_2\left(1 + \frac{1}{-1 + t_3(\ldots)}\right)}\right)}$$

where $t_k \in \mathbb{Z}^+$ and $t_k \geq 2$ for Pippenger continued fractions $1 \leq \xi \leq 2$.

ContinuedFraction:PositivePerronCaratheodoryContinuedFraction

Given a sequence of complex numbers d_n with $d_n \neq 0$ and $|d_n| < 1$, the positive Perron-Carathéodory continued fraction is the Perron-Carathéodory continued fraction

$$d_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{\begin{cases} -2\, d_0 & \text{for } n = 1 \\ 1 - \left| d_{(n-1)/2} \right|^2 z & \text{for } n > 1 \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases}}{\begin{cases} z & \text{for } n = 1 \\ \overline{d_{n/2}}\, z & \text{for } n \text{ even} \\ d_{(n-1)/2} & \text{for } n > 1 \text{ odd.} \end{cases}}$$

ContinuedFraction:PositiveThronFraction

A Thron fraction ξ of the form

$$\xi = \frac{F_1\, z}{1 + G_1\, z + \frac{z\, F_2}{1 + G_2\, z + \frac{z\, F_3}{1 + G_3\, z + \cdots}}}$$

is said to be a positive Thron fraction or a positive T-fraction if $F_m, G_m > 0$ for all m.

ContinuedFraction:RealJFraction

A J-fraction ξ of the form

$$\xi = \frac{a_1}{z + b_1 - \frac{a_2}{z + b_2 - \frac{a_3}{z + b_3 - \cdots}}}$$

is said to be a real J-fraction provided that $a_m > 0$ and that $b_m \in \mathbb{R}$ for $m = 1, 2, 3, \ldots$. Subtly, real J-fractions are connected to both C-fractions and modified S-fractions in the following way: A modified regular C-fraction ξ_C of the form

$$\xi_C = \frac{a_1}{z + \frac{a_2}{1 + \frac{a_3}{z + \cdots}}}$$

is a real J-fraction provided that it is also a modified S-fraction. As a result, many of the practical applications of C- and modified S-fractions are inherently applicable to real J-fractions as well.

ContinuedFraction:RegularCFraction

There are at least two distinct definitions for a regular C-fraction in reputable literature.

Some sources say that a C-fraction ξ of the form

$$\xi = b_0 + \mathop{\mathrm{K}}_{m=1}^{\infty} \frac{a_m z^{\alpha_m}}{1},$$

$a_m \in \mathbb{C}\backslash\{0\}$, is regular provided that $\alpha_m = 1$ for $m \geq 1$. Regular C-fractions which fall under this definition have strong connections to the theory of Padé approximations.

For the second definition, let $P(z) = c_0 + c_1 z + c_2 z^2 + \cdots$, $c_0 \neq 0$, be a formal power series with coefficients $c_k \in \mathbb{C}$ and let ξ be its corresponding continued fraction (i.e., its C-fraction) of the form

$$\xi = c_0 + \cfrac{a_1 z^{\alpha_1}}{1 + \cfrac{a_2 z^{\alpha_2}}{1 + \cfrac{a_3 z^{\alpha_3}}{1 + \cdots}}}$$

subject to the "correspondence relations"

$$(c_n, c_{n-1}, c_{n-2}, \ldots) \begin{pmatrix} \delta_{p,0} \\ \delta_{p,1} \\ \delta_{p,2} \\ \vdots \end{pmatrix} =$$

$$\begin{cases} 0 & \text{for } \alpha_0 + \cdots + \alpha_p < n < \alpha_1 + \cdots + \alpha_{p+1} \\ (-1)^p a_1 a_2 \cdots a_{p+1} & \text{for } n = \alpha_1 + \cdots + \alpha_{p+1}, \end{cases}$$

where $\delta_{i,j}$ denotes Kronecker's delta. Such a continued fraction is said to be regular if every approximant $\xi_p = A_p(z) / B_p(z)$, $n = 0, 1, 2, \ldots$, of ξ is a Padé approximant of $P(z)$.

There are a number of equivalent statements for the C-fraction ξ being regular by way of the second definition, many of which are more explicitly-stated than the above. For example, if $\xi_p = A_p(z) / B_p(z)$ denotes the pth approximant of ξ and if the numerator, respectively denominator, of ξ_p has degree s_p, respectively t_p, then ξ is regular if and only if there exists a sequence $\{r_n\}$ of natural numbers satisfying

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{p+1} = r_p + s_p + t_p + 1,$$

$p = 0, 1, 2, \ldots$. Moreover, this is equivalent to requiring that, for $p = 1, 2, 3, \ldots$,

$$\begin{aligned} s_{2p-1} &= \alpha_1 + \alpha_3 + \cdots + \alpha_{2p-1} \\ t_{2p} &= \alpha_2 + \alpha_4 + \cdots + \alpha_{2p}, \end{aligned}$$

while simultaneously requiring the existence of a sequence $\{k_n\}$ of integers which satisfy

$$\begin{aligned} s_{2p} &\leq t_{2p} + k_p \\ s_{2p+1} &\geq t_{2p+1} + k_p + 1 \\ s_{2p+1} &\geq t_{2p} + k_p + 1 \end{aligned}$$

$$s_{2p+1} \leq t_{2p+2} + k_p$$

for $p = 0, 1, 2, \ldots$.

As exemplified in the definition above, regular C-fractions are intimately connected to the study of formal power series and Padé approximants, as well as to the study of meromorphic complex-valued functions. Extensive exposition of this topic can be found in the works of H.S. Wall.

ContinuedFraction:Remainder

Let ξ be a real number with regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

(with M possibly ∞ for irrational numbers) with convergents A_n / B_n.

The nth remainder r_n of the continued fraction is defined through

$$\xi = b_0 + \underset{k=1}{\overset{n}{\mathrm{K}}} \frac{1}{\left(1 - \delta_{k,n}\right) b_k + r_n}.$$

The nth remainder (also called tail) r_n fulfills the following identities:

$$\xi = b_0 + \frac{A_{n-1}\, r_n + A_{n-2}}{B_{n-1}\, r_n + B_{n-2}}$$

$$\left|\xi - \frac{A_{n-1}}{B_{n-1}}\right| = \frac{1}{B_{n-1}(B_{n-1}\, r_n - B_{n-2})}.$$

ContinuedFraction:RiccatiSolution

In general, a Riccati differential equation is any first-order differential equation that is quadratic in the unknown function $y(x)$, and while there are a considerable number of differential equations attributed to Riccati, perhaps the most commonly agreed upon is the general equation

$$\frac{dy}{dx} = h(x) + g(x)\, y(x) + f(x)\, y^2(x)$$

where $f(x)$, $g(x)$, $h(x)$ are all continuous functions which are sufficiently differentiable for which $f(x), h(x) \neq 0$. Devised as a method to approximate solutions to differential equations of the form $y'(x) = f(x, y)$ by way of a second order Taylor approximation in y, a considerable number of solution techniques have been employed throughout the centuries, perhaps the most novel of which is the continued fraction solution first employed by Euler which has since been elaborated and expanded upon in great generality. A brief explanation of one such variant (stemming from Lagrange, as employed by Kurilin) follows.

Kurilin) follows.

Kurilin's method is based on approximating y by a sequence y_n which is specially defined depending upon how the zeroth approximation $y_0 = \xi_0$ is chosen. Once ξ_0 is defined, ξ_n (and hence y_n, which depends upon f_n, g_n, and h_n) is defined recursively by the relation

$y_{n-1} = \xi_{n-1}(x)\,[1 + y_n(x)]^{-1}$.

Finally, it follows from a simple analysis that the regular continued fraction ξ, defined to be the limit of the convergents $A_n/B_n = [\xi_0; \xi_1, \xi_2, \ldots, \xi_n]$ as $n \to \infty$, satisfies the generalized Riccati equation above.

Despite his solution being somewhat involved with a number of cases considered, the easiest and perhaps most illustrative of Kurilin's defined cases comes when $\xi_0 = \pm\sqrt{-h/f}$. In this case, one can prove that the nth approximation y_n of y satisfies $y_n' = f_n(x)\, y_n^2 + g_n(x)\, y_n + h_n(x)$, that $\xi_n(x)$ has the form

$$\xi_n(x) = \pm\left[\frac{\xi_{n-1}'(x) - g_{n-1}(x)\,\xi_{n-1}(x)}{h_{n-1}(x)}\right]^{1/2}$$

for $n \geq 2$ and that for $n \geq 2$, the remaining approximant functions f_n, g_n, h_n satisfy the recursions $f_n(x) = \xi_{n-1}(x)\, f_{n-1}(x)$,

$$h_n(x) = \frac{\xi_{n-1}'(x)}{\xi_{n-1}(x)} - h_{n-1}(x) - 2\,\frac{h_{n-2}(x)}{\xi_{n-2}(x)},$$

$$\frac{g_n = \frac{\xi_{n-1}'(x)}{\xi_{n-1}(x)} - g_{n-1}(x) - \frac{2}{\xi_{n-1}(x)}\,\xi_{n-2}'}{\xi_{n-2} + 2\, g_{n-2}(x).}$$

To complete the recurrence definition, one defines $g_0(x) = g(x)$, $f_0(x) = f(x)$, and $h_0(x) = h(x)$, and uses for $n = 1$ the equations

$$\xi_n(x) = \pm\left[-\frac{h_n(x)}{f_n(x)}\right]^{1/2},$$

$$g_n(x) = \frac{\xi_{n-1}'(x)}{\xi_{n-1}(x)} - g_{n-1}(x) - 2\,\frac{h_{n-1}(x)}{\xi_{n-1}(x)},$$

$f_n(x) = -h_{n-1}(x)/\xi_{n-1}(x)$, and $h_n(x) = -g_{n-1}(x) + \xi_{n-1}'(x)/\xi_{n-1}(x)$. A more detailed derivation can be found in the works of Chisholm.

ContinuedFraction:RogersRamanujanContinuedFraction

Given sequences of complex numbers a_n with $a_n \neq 0$ and complex q, the Rogers-Ramanujan continued fraction is the generalized continued fraction

$$q^{1/5} + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{q^n}{1}.$$

ContinuedFraction:RosenFraction

Let ξ be a real number. Then the Rosen continued fraction expansion for $q \in \mathbb{Z}^+$, $q \geq 3$, and $\lambda_q = 2\cos(\pi/q)$

$$\xi = \varepsilon_0\, b_0 + \underset{j=1}{\overset{N}{\mathrm{K}}} \frac{\varepsilon_j}{b_j}$$

(where N is possibly infinity), $\varepsilon_j \in \{-1, 1\}$, and $b_j \in \mathbb{Z}^+$ can be calculated through the repeated application of the map τ: $[-\lambda/2, \lambda/2 \to [\lambda/2, \lambda/2$

$$\tau(x) = \frac{\operatorname{sgn}(x)}{x} - \lambda \left\lfloor \frac{\operatorname{sgn}(x)}{\lambda\, x} + \frac{1}{2} \right\rfloor.$$

ContinuedFraction:RudinShapiroContinuedFraction

Let $r = \{r_n\}_{n=1}^{\infty}$ be a sequence whose nth term r_n is defined to be $+1$ if the number of occurrences of the string "11" in the binary representation of n is even and is defined to be -1 otherwise. The sequence r is called the Rudin-Shapiro sequence and the regular continued fraction $\xi = [0; r_0, r_1, r_2, \ldots]$ is called the Rudin-Shapiro fraction associated to r. This construction can be also generalized by way of the transformation $1 \mapsto a$, $-1 \mapsto b$ for distinct positive integers $a, b \in \mathbb{Z}^+$, whereby $r_k \in \{a, b\}$ for all $k = 0, 1, 2, \ldots$.

Unlike the similarly-defined Baum-Sweet fraction, the Rudin-Shapiro fraction is the focus of considerably more literature, having been generalized and applied to a variety of problems in areas such as polynomial theory, moment problems, and multiresolution analysis. Moreover, one of the more well-known properties of the Rudin-Shapiro fraction ξ is that it is transcendental, a result which can be proved by advanced numerical methods found, e.g., in the work of Adamczewski.

ContinuedFraction:SchmidtExpansion

The Schmidt expansion of a complex number $z = a + b\,i \in \mathbb{C}$, $b \geq 0$, assigns to z a complex continued fraction ξ of the form

$$\xi = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots}}}$$

whose successive approximants $\xi_n = A_n / B_n$ are determined by the Schmidt regular chain algorithm and whose elements a_k, b_k are Gaussian integers. The Schmidt expansion is an alternative to the more widely-utilized Hurwitz expansion and is known for assigning overall more accurate convergents to z despite having a lengthier and oftentimes slower computational implementation.

ContinuedFraction:SchurNevanlinnaFraction

Given a sequence of complex-valued functions $\{f_s\}_{s=0}^{\infty}$ which satisfy the recursive relation

$$f_{s+1}(z) = \frac{f_s(z) - f_s(0)}{1 - f_s(0)\, f_s(z)} \cdot \frac{1}{z}$$

for $s = 0, 1, 2, \ldots$, the associated Schur-Nevanlinna continued fraction ξ_0 for $f_0(z)$ has the form

$$\xi_0 = \frac{1}{f_0(0)} + \cfrac{c_0}{1 + d_0\, z + \cfrac{c_1\, z}{1 + d_1\, z + \cdots \cfrac{c_r\, z}{1 + d_r\, z + \cdots}}}.$$

Subsequent continued fractions ξ_s are formed by substituting ξ_0 into the aforementioned recursion.

ContinuedFraction:SemiConvergent

Let ξ be a real number with regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{M}{\mathrm{K}}} \frac{1}{b_k}$$

(for M possibly ∞) with convergents A_n / B_n.

A fraction p/q is called a best rational approximation of ξ if

$$\left|\xi - \frac{p}{q}\right| < \left|\xi - \frac{r}{s}\right|$$

for any integers r and s such that $s \leq q$ and $p/q \neq r/s$.

Every convergent A_n / B_n is best rational approximation of ξ, but not all best rational approximations are convergents of ξ.

The best rational approximation of ξ that are not convergents are called semi-convergents.

ContinuedFractionSemiConvergentRepresentation

Let ξ be a real number with regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{M}{\mathrm{K}}} \frac{1}{b_k}$$

(with M possibly ∞) with convergents A_n / B_n.

All semi-convergents $S_{n,g}$ of ξ are of the form

$$S_{n,g} = \frac{A_n + g\, A_{n+1}}{B_n + g\, B_{n+1}}$$

where $g \in \mathbb{Z}^+$ and

$$\left\lfloor \frac{b_{n+2}}{2} \right\rfloor < g < b_{n+2}$$

and potentially also for $g = \lfloor b_{n+2} / 2 \rfloor$.

Semi-convergents have the continued fraction expansion

$$S_{n,g} = b_0 + \underset{k=1}{\overset{M}{\mathrm{K}}} \frac{1}{\delta_{M,k}\, h + \left(1 - \delta_{M,k}\right) b_k}$$

where $M \geq 1$, $b_k > 1$ and $1 \leq h \leq b_k$.

ContinuedFraction:SequenceOfRightTails

Let ξ be a generalized continued fraction of the form

$$\xi = \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{b_m}{1},$$

$b_k \in \mathbb{C} \setminus \{0\}$, $k = 1, 2, 3, \ldots$, and suppose that ξ converges to some extended complex number $\alpha \in \hat{\mathbb{C}}$. Define $f^{(0)} = f$ and

$$f^{(n)} = \underset{m=n+1}{\overset{\infty}{\mathrm{K}}} \frac{b_m}{1}$$

for $n = 1, 2, 3, \ldots$, The sequence $\left\{f^{(n)}\right\}_{n=0}^{\infty}$ is called the sequence of right tails of ξ.

ContinuedFraction:SFraction

Consider the family of generalized continued fractions ξ_S which have the form

$$\xi_S = \frac{a_1 z}{1 + \frac{a_2 z}{1 + \frac{a_3 z}{1 + \cdots}}},$$

where the elements a_n are all strictly positive real numbers. Fractions of this form are called Stieltjes-fractions or S-fractions due to their prevalence in the work of Stieltjes and can be viewed as modifications of the other “named families” of continued fractions in several different ways. For example, ξ_S can be viewed as a C-fraction for which $b_0 = 0$, $a_n \in \mathbb{R}^+$, and $\alpha_n = 1$ for $n = 1, 2, \ldots$; at the same time, it can be considered as a modified Thron fraction with $F_n = a_n \in \mathbb{R}^+$ and with $G_n = 0$ for all n. From an application standpoint, the S-fraction is used in the theory of moment problems, as well as in the related study of formal power and Taylor series.

ContinuedFraction:SingularContinuedFraction

A continued fraction ξ of the form

$$\xi = b_0 + \underset{m=1}{\overset{N}{\mathrm{K}}} \frac{a_m}{b_m}$$

(here, N may be infinity) is said to be singular if for all $k \geq 1$, $b_k \geq 2$ and $b_k + a_k \geq 2$.

ContinuedFraction:Singularization

Singularization of a regular continued fraction is the removal of 1 ś in the partial denominators. Let the regular continued fraction of ξ have the j^{th} partial denominator with value $a_j = 1$

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2+\cfrac{1}{\dots+\cfrac{1}{a_{j-1}+\cfrac{1}{1+\cfrac{1}{a_{j+1}+\cfrac{1}{a_{j+2}+\cfrac{1}{\dots}}}}}}}},$$

then this 1 can be singularized to the new continued fraction

$$\xi = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2+\cfrac{1}{\dots+\cfrac{1}{\left(a_{j-1}+1\right)+\cfrac{1}{1-\cfrac{1}{a_{j+1}}+\cfrac{1}{a_{j+2}+\cfrac{1}{\dots}}}}}}}.$$

ContinuedFraction:SleszynskiPringsheimContinuedFraction

Given sequences of complex numbers a_n, b_n with for $n > 0$, $|b_n| > |a_n| + 1$, the Ślezyńsky-Pringsheim continued fraction is the generalized continued fraction

$$b_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n}.$$

ContinuedFractionsOfGeneralizedGaussMap

Let T_k, $k \in (-\infty, -1) \bigcup (0, \infty)$ be the generalized Gauss map

$$T_k(x) = \frac{1}{k\,\frac{x}{1-x}} - \left\lfloor \frac{1}{k\,\frac{x}{1-x}} \right\rfloor.$$

Then for some $\xi \in (0, 1)$ the generalized regular continued fraction expansion

$$\xi = \underset{j=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_j}$$

can be obtained through

$$b_j = T_k^j(\xi)$$

and inversely

$$\xi = \underset{j=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_j} = A_k^{-1}\, B^{b_1}\, A_k^{-1}\, B^{b_2} \ldots\ A_k^{-1}\, B^{b_n}\, A_k^{-1}(\infty)$$

where the maps A_k and B are defined through

$$A_k(x) = k\,\frac{x}{1-x}$$

$$B(x) = 1 + x.$$

ContinuedFractionsOfGeneralizedRenyiMap

Let T_k, $k \in (-\infty, -1) \bigcup (0, \infty)$ be the generalized Rényi map

$$T_k(x) = \frac{1}{k\,\frac{x}{1-x}} - \left\lfloor \frac{1}{k\,\frac{x}{1-x}} \right\rfloor.$$

Then for some $\xi \in (0, 1)$ the generalized regular continued fraction expansion

$$\xi = \underset{j=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_j}$$

can be obtained through

$$b_j = T_k^j(\xi)$$

and inversely

$$\xi = \underset{j=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_j} = A_k^{-1}\, B^{b_1}\, A_k^{-1}\, B^{b_2} \ldots\ A_k^{-1}\, B^{b_n}\, A_k^{-1}(\infty)$$

where the maps A_k and B are defined through

$$A_k(x) = k\,\frac{x}{1-x}$$

$$B(x) = 1 + x.$$

ContinuedFractionsWithGivenConvergents

Let A_n, B_n for $n = 0, 1, \ldots$ be two given sequences with

$B_0 = 1$

$A_n B_{n-1} - A_{n-1} B_n \neq 0.$

Then the continued fraction

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

will have the convergents p_n / q_n if

$b_0 = A_0$

$a_1 = A_1 B_0 - A_0 B_1$

$b_1 = B_1$

$$a_k = \frac{A_{n-1} B_n - A_n B_{n-1}}{A_{n-1} B_{n-2} - B_{n-2} B_{n-1}}$$

$$b_k = \frac{A_n B_{n-2} - A_{n-2} B_n}{A_{n-1} B_{n-2} - A_{n-2} B_{n-1}}.$$

ContinuedFraction:SymplecticContinuedFraction

Let M and J be block matrices of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

whose entries themselves are square matrices, I the square identity matrix of appropriate dimension. Given a collection $\{M_m, M_{m+1}, M_{m+2}, \ldots\}$ of matrices satisfying $M_k^T J M_k = J$ for all $k = m, m+1, m+2, \ldots$, a symplectic continued fraction is defined to be the sequence of formal approximants $\{T_{M_m M_{m+1} \cdots M_n}(\infty)\}$, $n = m, m+1, m+2, \ldots$, where $T_M(Z)$ is the (matrix) M bius transformation of the form

$T_M(Z) = (A Z + B)(C Z + D)^{-1}$

and where $T_M(\infty) = A C^{-1}$ by definition.

ContinuedFraction:TauFraction

Let $\phi = \left(1 + \sqrt{5}\right)/2$ denote the golden ratio. Then if $b_0 \in \mathbb{Z}$ is an arbitrary integer, if the sequence b_n is a collection of positive integers for $n = 1, 2, 3, \ldots$, and if $\epsilon_n \in \{\pm 1\}$ for each n, then the generalized continued fraction ξ_τ of the form

$$\xi_\tau = b_0 + \frac{\epsilon_1}{\phi\, b_1 + \frac{\epsilon_2}{\phi\, b_2 + \frac{\epsilon_3}{\phi\, b_3 + \cdots}}}$$

is said to be a τ-fraction. τ-fractions are a regular part of the study of algebraic number fields, particularly the one generated by $\phi = 2 \cos(\pi/5)$.

The fraction gets its name from the golden ratio ϕ, which is sometimes also denoted τ.

ContinuedFraction:TFraction

A generalized continued fraction ξ_T of the form

$$\xi_T = \frac{z}{1 + G_1\, z + \frac{z}{1 + G_2\, z + \frac{z}{1 + G_3\, z + \cdots}}}$$

for G_n a complex sequence, $n = 1, 2, 3, \ldots$, is called a T-fraction. Note, in particular, that T-fractions are specialized versions of the more general Thron fraction which result from setting $F_n = 1$ for all $n = 1, 2, 3, \ldots$.

ContinuedFraction:ThieleFraction

The so-called Thiele fraction is a generalized continued fraction $\xi_{a\,p\,p\,r}$ of the form

$$\xi_{\text{appr}} = b_0 + \frac{z - z_0}{b_1 + \frac{z-z_1}{b_2+\frac{z-z_2}{b_3+\cdots}}}$$

where here, the elements z_n and b_n are specially-chosen complex numbers defined as follows. Given a function f whose values are known at a collection $\{z_0, z_1, \ldots\}$ of distinct points, $z_n \in \mathbb{C}$, the collection of inverse differences $\varphi[z_1, \ldots, z_k]$ for $f(z)$ are formed using the recursive formulas:

- $\varphi_0[z_k] = f(z_k)$, $k \geq 0$.
- $\varphi_1[z_k, z_\ell] = \frac{z_\ell - z_k}{\varphi_0[z_\ell] - \varphi_0[z_k]}$, $\ell > k \geq 0$.
- $\varphi_\ell[z_0, \ldots, z_\ell] = \frac{z_\ell - z_{\ell-1}}{\varphi_{\ell-1}[z_0, \ldots, z_{\ell-2}, z_\ell] - \varphi_{\ell-1}[z_0, \ldots, z_{\ell-1}]}$, $\ell \geq 1$.

The Thiele fraction ξ_{appr} was defined as part of Thiele$\text{\char"0315}$s work on approximation theory and utilizes the collection $\{z_0, \ldots, z_n\}$ in two ways, both explicitly in its partial numerators and implicitly by defining $b_0 = \varphi_0[z_0]$ and $b_k = \varphi_k[z_0, \ldots, z_k]$ for $k = 1, 2, \ldots$. In this way, the fraction $\xi_{\text{appr}} = \xi_{\text{appr}}(z)$ is easily seen to be an interpolating function for $f(z)$ and as such has a wide variety of uses in the approximation theory of arbitrary complex-valued functions.

ContinuedFraction:ThreePeriodicFraction

A general continued fraction $\xi = b_0 + \mathrm{K}(a_m / b_m)$ is said to be three-periodic if the sequences $\{a_m\}$ and $\{b_m\}$ are three-periodic after the first N elements, i.e., if $a_{N+3\,p+q} = a_{N+q}$ and $b_{N+3\,p+q} = b_{N+q}$ where $N \in \mathbb{Z}^+$ fixed, $p \geq 1$, k is a fixed positive integer, and $q \in \{1, 2, 3, \ldots, k\}$. Explicitly, then, a k-periodic fraction ξ has the form

$$\xi = b_0 + \frac{a_1}{b_1 +}\,\frac{a_2}{b_2 +}\cdots\frac{a_N}{b_N +}\,\frac{a_{N+1}}{b_{N+1} +}\,\frac{a_{N+2}}{b_{N+2} +}\,\frac{a_{N+3}}{b_{N+3} +}\,\frac{a_{N+1}}{b_{N+1} +}\,\frac{a_{N+2}}{b_{N+2} +}\,\frac{a_{N+3}}{b_{N+3} +}\cdots$$

for some fixed positive integer N.

Worth noting is that a 3-periodic continued fraction is just a special case of a so-called k-periodic continued fraction (for $k = 3$) where k-periodicity means that $a_{N+k\,p+q} = a_{N+q}$ and $b_{N+k\,p+q} = b_{N+q}$ for $N \in \mathbb{Z}^+$ fixed, $p \geq 1$, k is a fixed positive integer, and $q \in \{1, 2, 3\}$. k-periodicity plays a significant role, e.g., in studying continued fraction convergence, in particular the study of convergence by way of tail sequence analysis. Such ideas are explored in greater detail in the works of Lorentzen and Waadeland.

ContinuedFraction:ThreeTermRecurrenceMinimalSolution

A non-trivial solution $\{f_n\}_{n=-1}^{\infty}$ of a three-term recurrence relation

$X_n = b_n X_{n-1} + a_n X_{n-2}$,

$a_n, b_n \in \mathbb{C}$ for $n = 1, 2, 3, \ldots$, $a_k \neq 0$ for all k, is said to be minimal if for any other solution $\{g_n\}$,

$$\lim_{n\to\infty} \frac{f_n}{g_n} = 0.$$

A general three-term recurrence relation may or may not have a minimal solution, and any non-minimal solution is said to be dominant.

A number of significant theorems pertaining to minimal solutions of recurrence relations hinge on the theory of continued fractions. For example, Pincherle proved that for sequences $\{a_n\}$ and $\{b_n\}$ of a normed field $\mathbb{F}$ (with $a_n \neq 0$ for $n = 1, 2, 3, \ldots$), the three-term recurrence relation

$X_n = b_n X_{n-1} + a_n X_{n-2}$

has a minimal solution $\{h_n\}$, $h_n \in \mathbb{F}$ for all n, if and only if the associated continued fraction ξ of the form

$$\xi = \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{a_m}{b_m}$$

converges in $\mathrm{F} \bigcup \{\infty\}$ and, moreover, that such a solution satisfies the associated continued fraction relation

$$\frac{h_m}{h_{m-1}} = \frac{-a_m}{b_m} + \underset{n=m+1}{\overset{\infty}{K}} \frac{a_n}{b_n}$$

for all m. A considerable amount of information concerning the role of continued fractions in three-term recurrence relations and minimal solutions thereto can be found in the works of Pincherle and Gautschi.

ContinuedFraction:ThreeTermRecurrenceSolution

A sequence $\{X_n\}_{n=-1}^{\infty}$ of complex numbers is a solution of the three-term recurrence relation

$$X_n = b_n X_{n-1} + a_n X_{n-2}$$

provided that all consecutive triples of its elements are solutions. Here, $a_n, b_n \in \mathbb{C}$ for $n = 1, 2, 3, \ldots$ and $a_k \neq 0$ for all k. A well-known fact in the study of continued fractions is that the approximants $\xi_n = A_n / B_n$ of an arbitrary continued fraction ξ satisfy the three-term recurrence relation with the initial conditions $A_{-1} = 1$, $A_0 = b_0$, $B_{-1} = 0$, and $B_0 = 1$.

Continued fractions are connected to the three-term recurrence relation at an even deeper level as well. For example, one can show that the solution space for the three-term recurrence relation is a linear space $\mathfrak{L}$ of dimension 2 over $\mathbb{C}$ and that the canonical numerators and denominators $\{A_n\}$ and $\{B_n\}$ of $K(a_n / b_n)$ actually form a basis for $\mathfrak{L}$. It can also be shown that the recurrence relation has a so-called minimal solution precisely when the continued fraction

$$\xi = \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{a_m}{b_m}$$

converges in $\hat{\mathbb{C}}$. A considerable amount of information concerning the role of continued fractions in three-term recurrence relations and minimal solutions thereto can be found in the works of Pincherle and Gautschi.

ContinuedFraction:ThronFraction

Consider a generalized continued fraction ξ_{Th} of the form

$$\xi_{\mathrm{Th}} = \frac{F_1\, z}{1 + G_1\, z + \frac{z\, F_2}{1 + G_2\, z + \frac{z\, F_3}{1 + G_3\, z + \cdots}}}$$

where F_n, G_n are sequences of complex numbers and $F_k \neq 0$ for $k = 1, 2, \ldots$. Such fractions are called generalized T-fractions or Thron fractions after mathematician Wolfgang Thron. In an obvious way, Thron fractions are generalizations of the M-fraction obtained by replacing $F_1 \neq 0$ in a standard M-fraction ξ_M with $F_1\, z$. In addition, Thron fractions are often further classified based on properties of the elements F_n, G_n of ξ_{Th}. For example:

- Thron fractions ξ_T for which $F_n = 1$, $n = 1, 2, 3, \ldots$, are called T-fractions.
- Thron fractions which satisfy $F_m, G_m > 0$ for all m are called positive T-fractions.
- Thron fractions for which $F_m, G_m \in \mathbb{R} \setminus \{0\}$ and which satisfy the conditions $F_{2m-1}\, F_{2m} > 0$, $F_{2m-1} / G_{2m-1} > 0$ are called alternating positive term fractions or APT-fractions.

Unsurprisingly, as the expansive classification suggests, the applications of Thron fractions are also large in number. In the same way that C-, J-, and M-fractions play crucial roles in the understanding of formal power series, for example, Thron fractions— and in particular, T-fractions— are critical tools used in the study of formal Taylor series. Like the above-mentioned M-fractions, Thron fractions and the offshoots thereof correspond to expansions of these formal Taylor series at two points. For more information concerning the variety of Thron fractions as well as other continued fraction results from Thron's extensive work.

ContinuedFraction:TwoDimensional

A two-dimensional continued fraction is an expression of the form

$$\xi(x,\, y) = B_0 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{a_j\, x\, y}{B_j}$$

where

$$B_j = b_0^{(j)} + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{c_k^{(j)}\, x}{1} + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{d_k^{(j)}\, y}{1}.$$

ConvergenceDomainOfTFractions

Let ξ be a Thiele fraction with periodic limits,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n z}{1 + b_n z}$$

$$\lim_{n\to\infty} a_{l+m\,n} = a^l$$

$$\lim_{n\to\infty} b_{l+m\,n} = b^l.$$

Let

$$D = \mathbb{C} - (\Gamma \bigcup K)$$

be a domain for f which is a meromorphic function with poles V in D. Let K be a finite set and X be any compact set in D disjoint from V. Let

$$E = \left[0,\ 4\,(-1)^m \prod_{l=1}^{m} a^l\right]$$

be a real interval, Γ be defined by

$$\Gamma = \{z \mid z^{-m} \left(\prod_{i=1}^{m} \begin{pmatrix} 0 & z\,a^i \\ 1 & 1 + z\,b^i \end{pmatrix}\right)^2 \in E,$$

$$D_1 = \mathbb{C} - \Gamma$$

be a domain, $D_{z_0}(\epsilon)$ be a disk with center z_0 in Γ of radius ϵ, and

$$D_2 = D_1 \bigcup D_{z_0}(\epsilon)$$

be a domain. Then

$\forall_X$, ξ converges to f in X,

the number of elements in $K \le (-1 + m)\, m$,

if $b^l = 0$, then the number of elements in $K \le \lfloor m\,(m-1)/2 \rfloor$,

f has a meromorphic continuation to D_1 and it has no continuation to D_2 for any choice of z_0 and ϵ.

ConvergenceOfConstantNumeratorContinuedFraction

Let ξ be the continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{\delta_{k,1} + \left(1 - \delta_{k,1}\right) c}{1}.$$

Then ξ converges for $c \in \mathbb{C} \backslash (-\infty, -1/4)$.

ConvergenceOfDiagonalPadeApproximantsForAnalyticFunctionsWithFiniteNumberOfBranchPoints

Let f be a multivalued holomorphic function, Σ be a finite set where $\Sigma \subset \mathbb{C}$,

$\Omega = [\text{Riemann sphere}] - \Sigma$

be the domain of f, g be an analytic continuation of f at infinity on a domain D, R_n be the Padé approximants diagonal for f; then there is a unique domain $D \subset \Omega$ that is maximal by inclusion among domains where R_n converges in capacity on compact sets to a single-valued g on D.

ConvergenceOfEllipticContinuedFractions

An arbitrary p-periodic elliptic continued fraction $\xi = \mathrm{K}(a_n/b_n)$ diverges generally, and because convergence in the classical sense implies convergence in the general sense, ξ elliptic also fails to converge classically. The statement of this fact can be found in the work, e.g., of Lorentzen and Waadeland and can be justified by noting that the sequence $\{S_n(\xi)\}$ corresponding to an elliptic continued fraction ξ is totally non-restrained where here, S_n is the Möbius transformation defined for all $w \in \mathbb{C}$ by the approximant function

$$S_n(w) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}.$$

Worth noting is that requiring an elliptic continued fraction to satisfy additional criteria may indeed alter its convergence behavior. For example, the construction of an elliptic limit 1-periodic continued fraction which converges can be found in the works of Gill, who also gives classification criteria for the convergence of limit-periodic continued fractions based on the relative convergence rates of the $(n\,p)$th tail of $S_n(\xi)$.

ConvergenceOfIdentityTypeContinuedFractions

An arbitrary p-periodic identity-type continued fraction $\xi = \mathrm{K}(a_n/b_n)$ diverges generally, and because convergence in the classical sense implies convergence in the general sense, ξ elliptic also fails to converge classically. The statement of this fact can be found in the work, e.g., of Lorentzen and Waadeland and can be justified by noting that the sequence $\{S_n(\xi)\}$ corresponding to an identity-type continued fraction ξ is totally non-restrained where here, S_n is the M bius transformation defined for all $w \in \mathbb{C}$ by the approximant function

$$S_n(w) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}.$$

Worth noting is that the aforementioned convergence properties of identity-type continued fractions identically mimic those for elliptic fractions. Unlike elliptic fractions whose convergence behavior can be altered by enforcing additional criteria, the literature mentions no such alterations for identity-type fractions.

ConvergenceOfLimitPeriodicContinuedFractions

The limit periodic continued fraction $\xi = \mathrm{K}(1/b_n) = [0; b_1, b_2, \ldots]$ converges to $b = 1/4$ if $|b_n - (-1/4)| < 1/4\left(4 n^2 - 1\right)$ for all $n = 1, 2, \ldots$.

ConvergenceOfLoxodromicContinuedFractions

An arbitrary p-periodic loxodromic continued fraction $\xi = \mathrm{K}(a_n/b_n)$ converges in the general sense to one of the two fixed points of the sequence $\{S_p\}$, namely to its attracting fixed point $x \in \mathbb{C} \bigcup \{\infty\}$. On the other hand, if y denotes the repelling fixed point of the sequence S_n, then ξ is guaranteed to converge in the classical sense if and only if $S_k(0) \neq y$ for all $k = 1, 2, 3, \ldots$. Here, S_n is the Möbius transformation defined for all $w \in \mathbb{C}$ by the approximant function

$$S_n(w) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}.$$

Because they are only conditionally convergent in the classical sense, loxo-dromic fractions fail to converge uniformly in any nontrivial metric. In addition, because of the "closeness" with which loxodromic fractions ξ are related to the parabolic fractions, a seemingly unpredictable pattern of convergent behavior is obtained by implementing stricter structural rules on ξ. For example, for ξ limit p-periodic, one generally has to consider the value p as well as the speed with which the elements a_n, b_n of ξ converge to their respective limits. Such details are covered in more depth in the works of Lorentzen and Waadeland and its references.

ConvergenceOfPadeApproximantsForExponentialFunction

Let

$$f(z) = e^z$$

and $R_{n,m}(z)$ be its Padé approximants and let p_i and q_i be the subsequences.

Then given

$$\lim_{i \to \infty} (p_i + q_i) = \infty,$$

it follows that

$$\lim_{i \to \infty} R_{p_i,q_i}(z) = f(z).$$

ConvergenceOfParabolicContinuedFractions

An arbitrary p-periodic parabolic continued fraction $\xi = \mathrm{K}(a_n/b_n)$ converges in the general sense to the single fixed point x of the sequence $\{S_p\}$. Moreover, because ξ parabolic if and only if the sequence $\{S_p\}$ is and because $\{S_p(w)\}$ can be shown to converge to x for every $w \in \mathbb{C}$, one can easily conclude by way of analyzing its tail-values that ξ also converges to x in the classical sense. Here, S_n is the Möbius transformation defined for all $w \in \mathbb{C}$ by the approximant function

$$S_n(w) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}.$$

Lorentzen and Waadeland point out that despite their apparent good behavior, parabolic continued fractions fail to converge according to other, more stringent definitions. For example, one can show that ξ parabolic still fails to converge uniformly to x in $\mathbb{C} \cup \{\infty\}$, even when the metric considered is the chordal metric. In addition, because of the "closeness" with which parabolic fractions ξ are related to the always-divergent elliptic fractions, a seemingly unpredictable pattern of convergence behavior is obtained by implementing stricter structural rules on ξ. For example, for ξ limit p-periodic, one generally has to consider the value p as well as the speed with which the elements a_n, b_n of ξ converge to their respective limits. Such details are covered in more depth in the works of Lorentzen and Waadeland and its references.

ConvergenceOfRogersRamanujanContinuedFractionAtPrimitiveRootsOfUnity

Let $R(q)$ be the Rogers-Ramanujan continued fraction and $K(q)$ be

$$K(q) = \frac{q^{1/5}}{R(q)}.$$

Then there exists an uncountable constructible set $G \subset \{z : |z| \leq 1\}$ such that $K(y)$ does not converge generally for all $y \in G$.

ConvergenceRadiusOfPadeApproximantRows

Let f be a meromorphic function, and $D(m)$ be the largest complex disk where f has less than or equal to m poles, and $d(m)$ be the divisor of its poles.
Let $T_{m,n}$ be the mth row Padé approximants, R_m be the radius of $D(m)$, a be an element of $\mathbb{C} - 0$, $U(a)$ be the poles converging from $T_{m,n}$ at a,
$\mu(a) = \text{the number of elements in } U(a)$
$Q_{n,m}$ be the Padé approximants denominators and $Q^*_{n,m}$ be the spherical normalizations of $Q_{n,m}$,
$\Delta(a) = \limsup_{n\to\infty} \left| \left(Q^*_{n,m}\right)(a) \right|^{1/n}$,
P_m be the set where $a\mu(a) \geq 1$, and
$E_m = \{\{a, \mu(a)\} \mid a \in P_m\}$
be a divisor. Then
$\forall_{a \in P_m} R_m = \dfrac{|a|}{\Delta(a)}$
$d(m) = e_m$.

ConvergenceRadiusOfPadeApproximantRowsWithMPoles

Let f be a meromorphic function, and $D(m)$ be the largest complex disk where f has less than or equal to m poles. Let $T_{m,n}$ be the m th row Padé approximants. If the number of poles in $D(m)$ is exactly m, then $T_{m,n}$ converges to f in the chordal metric on the Riemann sphere.

ConvergenceSetBoundednessForComplexContinuedFractionProducts

For any set E of complex numbers, denote by $V_B(E)$ the set of all finite continued fractions $\underset{i=1}{\overset{n}{\mathrm{K}}}\, 1/b_i$ with elements $b_m \in E$. Call a set S a convergence set of type B if $b_m \in E$ for all $m \geq 1$ ensures the convergence of $\underset{i=1}{\overset{n}{\mathrm{K}}}\, 1/b_i$. Then if $z = -1$ is not a limit point of $W_B(E) = \{u \cdot v : u \in V_B(E),\ v \in V_B(E)\}$, E is a convergence set of type B for $\underset{i=1}{\overset{n}{\mathrm{K}}}\, 1/b_i$ if and only if $V_B(E)$ is bounded.

ConvergenceSetBoundednessForComplexContinuedFractio

nSums

For any set E of complex numbers, denote by $V_A(E)$ the set of all finite continued fractions $\mathop{\mathrm{K}}_{i=1}^{n} a_i/1$ with elements $a_m \in E$. Call a set S a convergence set of type A if $a_m \in E$ for all $m \geq 1$ ensures the convergence of $\mathop{\mathrm{K}}_{i=1}^{n} a_i/1$. Then if $z = -1$ is not a limit point of $W_A(E) = \{u + v : u \in V_A(E), v \in V_A(E)\}$, E is a convergence set of type A for $\mathop{\mathrm{K}}_{i=1}^{n} a_i/1$ if and only if $V_A(E)$ is bounded.

ConvergenceSetBoundednessForRealContinuedFractionProducts

For any set E of real numbers, denote by $V_B(E)$ the set of all finite continued fractions $\mathop{\mathrm{K}}_{i=1}^{n} 1/b_i$ with elements $b_m \in E$. Call a set S a convergence set of type B if $b_m \in E$ for all $m \geq 1$ ensures the convergence of $\mathop{\mathrm{K}}_{i=1}^{n} 1/b_i$. Then E is a convergence set of type B for $\mathop{\mathrm{K}}_{i=1}^{n} 1/b_i$ if and only if $V_B(E)$ is bounded.

ConvergenceSetBoundednessForRealContinuedFractionSums

For any set E of real numbers, denote by $V_A(E)$ the set of all finite continued fractions $\mathop{\mathrm{K}}_{i=1}^{n} a_i/1$ with elements $a_m \in E$. Call a set S a convergence set of type A if $a_m \in E$ for all $m \geq 1$ ensures the convergence of $\mathop{\mathrm{K}}_{i=1}^{n} a_i/1$. Then E is a convergence set of type A for $\mathop{\mathrm{K}}_{i=1}^{n} a_i/1$ if and only if $V_A(E)$ is bounded.

ConvergenceTheoremForPeriodicIntegralContinuedFractionsWithVariableUpperLimits

Let

$$K(t) = \underset{k=1}{\overset{\infty}{D}} \int_{t_0}^{\tau_{k-1}} \frac{a(\tau_k - 1, \tau_k)}{b(\tau_k - 1, \tau_k)} \, d\tau_k$$

be a periodic integral continued fraction, τ_k be the periodic integral continued fraction integration limit set of $K(t)$, $a(\tau, \xi)$ and $b(\tau, \xi)$ be continuous complex-valued functions on the domain $\Omega = [t_0, T] \times [t_0, \tau]$, and $A_r(t)$ be the rth convergent.

Write τ^k for $(\tau_k - 1, \tau_k)$ and set

$$Q_{k,n}(\tau^k) = \begin{cases} b(\tau^n) & \text{for } k = n \\ b(\tau^k) + \int_{t_0}^{\tau_{k-1}} \frac{a(\tau^{k+1})}{Q(k+1,n)(\tau^{k+1})} \, d\tau_{k+1} & \text{for } 1 \leq k \wedge k < n. \end{cases}$$

Then given $g(\tau, \xi)$ is a continuous function such that $\left|Q_{k,n}(\tau^k)\right| \geq g(\tau, \xi)$, $K(t)$ converges absolutely and uniformly and

$|K(t) - A_r(t)| \leq \frac{m^{-2r-1} M^{r+1} (t-t_0)^{r+1}}{(r+1)!}$,

where

$M = \max(t_0 \leq \xi \leq \tau \leq T, |a(\tau, \xi)|)$

$m = \min(t_0 \leq \xi \leq \tau \leq T, |g(\tau, \xi)|)$.

ConvergenceTheoremForSequenceOfEvenApproximants

Let a be an arbitrary complex number and let $\rho > |a|$, $\rho \geq |a+1|$, and $\epsilon > 0$. Let the elements b_n of the continued fraction $\xi = [1; b_1, b_2, \ldots]$ satisfy

$$\begin{cases} b_{2n-1} = c_{2n-1}^2 & \text{for } |c_{2n-1} \pm i\, a| \leq \rho \\ b_{2n-1} = c_{2n-1}^2 & \text{for } |c_{2n} \pm i(a+1)| \geq \rho \end{cases}$$

and $|b_{2n}| \geq -|a+1|^2 + \rho^2 + \epsilon$. Then the even part of ξ converges to a value ν which satisfies $|\nu - (a+1)| \leq \rho$.

ConvergentsDenominatorGrowth

Let $0 < \xi < 1$ be an irrational number with regular continued fraction representation

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

and A_n / B_n the sequence of its convergents.

Then for almost all ξ and any $\varepsilon > 0$ the following identity holds as $n \to \infty$:

$$\sqrt[n]{B_n} = \exp\left(\frac{\pi^2}{12 \ln(2)}\right) + o\left(\frac{1}{\sqrt{n}} \ln^{(3+\varepsilon)/2}(n)\right).$$

ConvergentsDenominatorGrowthBound

Let ξ be a regular continued fraction, B_n be the convergent denominator of ξ, and F_n be the Fibonacci sequence. Then $B_n \geq F_n$.

ConvergentsIrreducibility

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction with indeterminates a_k, b_k and p_k / q_k the sequence of its convergents.

Then for all $n \in \mathbb{Z}^+$, the convergents numerators $p_k(a_1, a_2, \ldots, a_n, b_0, b_1, b_2, \ldots, b_n)$ and denominators $q_k(a_1, a_2, \ldots, a_n, b_0, b_1, b_2, \ldots, b_n)$ as polynomials in the indeterminates a_k, b_k are irreducible polynomials.

ConvergentsMatrixRepresentations

Let $0 < \xi < 1$ be a regular continued fraction

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

and A_n / B_n the sequence of its convergents.

Then the following representations for the convergents holds:

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \left(\prod_{k=1}^{n} \begin{pmatrix} 1 & 0 \\ 0 & b_k \end{pmatrix}\right) . \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

ConvergentsNumeratorAndDenominatorRelativelyPrime

Let ξ be a generalized continued fraction, A_n be the convergent numerator of ξ, and B_n be the convergent denominator of ξ. Then

$\gcd(A_n, B_n) = 1$.

ConvergentsNumeratorGrowth

Let $0 < \xi < 1$ be an irrational number with regular continued fraction representation

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

be a continued fraction and A_n / B_n the sequence of its convergents.

Then for almost all ξ and any $\varepsilon > 0$ the following identity holds as $n \to \infty$:

$$\sqrt[n]{A_n} = \sqrt[n]{\xi} \exp\left(\frac{\pi^2}{12 \ln(2)}\right) + o\left(\frac{1}{\sqrt{n}} \ln^{(3+\varepsilon)/2}(n)\right).$$

ConvergentsOfCFractionsAreIrredicubleRationalFunctions

The nth convergent $A_n(x) / B_n(x)$ of a corresponding type continued fraction ξ of the form

$$\xi = 1 + \frac{b_1 x^{\alpha_1}}{1 + \frac{b_2 x^{\alpha_2}}{1 + \frac{b_3 x^{\alpha_3}}{1 + \cdots}}}$$

is an irreducible rational fraction.

ConvergentsOfInverseRegularContinuedFraction

Let ξ be a regular continued fraction

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}$$

with $b_k \in \mathbb{Z}^+$ and A_k / B_k the sequence of its convergents.

Then for all $M \in \mathbb{Z}^+$ the following identities hold:

$$\frac{A_M}{A_{M-1}} = b_M + \underset{k=1}{\overset{M}{\mathrm{K}}} \frac{1}{b_{M-k}}$$

$$\frac{B_M}{B_{M-1}} = b_M + \underset{k=1}{\overset{M-1}{\mathrm{K}}} \frac{1}{b_{M-k}}.$$

CorollaryForMeromorphicExtensionOfJFractions1

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z+b_1-\cfrac{a_2}{z+b_2-\cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$, and

$$\sum_{j=1}^{\infty}\left(\left|\frac{a_j - 1}{4}\right| + \left|b_j\right|\right) < \infty.$$

For notational convenience, let $u_j = 2\, b_j$ for $j \geq 0$ and let $v_j = 1 - 4\, a_j$ for $j \geq 1$. If, for arbitrary $\omega \in \mathbb{C}$ with $w = \omega^2$, $C_n(\omega)$, $D_n(\omega)$ are terms which satisfy the recursions $C_0(\omega) = D_{-1}(\omega) = 0$, $C_1(\omega) = D_0(\omega) = 1 - w$, and

$C_{n+1}(\omega) - C_n(\omega) = w\,(C_n(\omega) - C_{n-1}(\omega)) + u_n\, \omega\, C_n(\omega) + v_n\, w\, C_{n-1}(\omega)$, for $n \geq 1$,

$D_{n+1}(\omega) - D_n(\omega) = w\,(D_n(\omega) - D_{n-1}(\omega)) + u_n\, \omega\, D_n(\omega) + v_n\, w\, D_{n-1}(\omega)$, for $n \geq 0$,

and if expressions $C(\omega)$, $D(\omega)$ are defined so that $C(\omega) = S_0(\omega)$, $D = S_{-1}(\omega)$ where

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega)$$

for

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\, u_j\left(1 - w^{j-k}\right) + w\, v_j\left(1 - w^{j-k-1}\right)\right)$$

with $c_{k,j}(\pm 1) = \pm (j-k)\, u_j + (j-k-1)\, v_j$ by definition, then the following hold:

1. For every fixed $0 < t < 1$, $\lim C_n(\omega) = C(\omega)$ and $\lim D_n(\omega) = D(\omega)$ uniformly for $|\omega| \leq t$.

2. The functions $C(\omega)$, $D(\omega)$ are holomorphic for $|\omega| < 1$, are continuous for $|\omega| \leq 1$, $\omega \neq \pm 1$, and satisfy $C \not\equiv 0$, $D \not\equiv 0$ due to the fact, e.g., that $C(0) = D(0) = 1$.

CorollaryForMeromorphicExtensionOfJFractions2

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z+b_1-\cfrac{a_2}{z+b_2-\cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$, and

$$\sum_{j=1}^{\infty}\left(\left|\frac{a_j - 1}{4}\right| + \left|b_j\right|\right) < \infty.$$

Furthermore, let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$ and for notational convenience. let $u_i = 2\, b_i$ for $i \geq 0$ and let $v_i = 1 - 4\, a_i$ for $i \geq 1$.

1. Uniformly on compact subsets of $|\omega| = 1$, $\omega \neq 1$,

$C_n(\omega) = C(\omega) - w^n\, C(\overline{\omega}) + O(1)$,

$D_n(\omega) = D(\omega) - w^{n+1}\, D(\overline{\omega}) + O(1)$

as $n \to \infty$, where $C_n(\omega)$, $D_n(\omega)$ are terms which satisfy the recursions
$C_0(\omega) = D_{-1}(\omega) = 0$, $C_1(\omega) = D_0(\omega) = 1 - w$,

$C_{n+1}(\omega) - C_n(\omega) = w\,(C_n(\omega) - C_{n-1}(\omega)) + u_n\, \omega\, C_n(\omega) + v_n\, w\, C_{n-1}(\omega)$, $n \geq 1$,

$D_{n+1}(\omega) - D_n(\omega) = w\,(D_n(\omega) - D_{n-1}(\omega)) + u_n\, \omega\, D_n(\omega) + v_n\, w\, D_{n-1}(\omega)$, $n \geq 0$,

and where $C(\omega) = S_0(\omega)$, $D = S_{-1}(\omega)$ for

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\, u_j \left(1 - w^{j-k}\right) + w\, v_j \left(1 - w^{j-k-1}\right)\right),$$

with $c_{k,j}(\pm 1) = \pm (j-k)\, u_j + (j-k-1)\, v_j$ by definition.

2. If in addition to the hypotheses in (1.) $\sum_{j=1}^{\infty} j\left(\left|a_j - 1/4\right| + \left|b_j\right|\right) < \infty$, then C and D are continuous for $|\omega| \leq 1$ and the asymptotic estimates for C_n, D_n in (1.) hold uniformly for all $|\omega| = 1$.

3. For all $|\omega| = 1$, $\omega \neq \pm 1$,

$$\omega^{-1}\, C\left(\omega^{-1}\right) D(\omega) - \omega\, C(\omega)\, D\left(\omega^{-1}\right) = \left(\omega^{-1} - \omega\right) \prod_{j=1}^{\infty} \left(1 - v_j\right).$$

4. For fixed $|\omega| = 1$, $\omega \neq \pm 1$, $\lim_{n\to\infty} C_n(\omega)$, respectively $\lim_{n\to\infty} D_n(\omega)$, exists and equals $C(\omega) \neq 0$, respectively $D(\omega) \neq 0$, if and only if $C(\overline{\omega}) = 0$, respectively $D(\overline{\omega}) = 0$. Moreover, at least one of the sequences $C_n(\omega)$, $D_n(\omega)$ diverges to ∞.

5. If all a_n, b_n the continued fraction expansion of $f(z)$ is real, then $C(\omega) = \overline{C(\overline{\omega})} \neq 0$ and $D(\omega) = \overline{D(\overline{\omega})} \neq 0$ both hold for all $|\omega| = 1$, $\omega \neq \pm 1$. In this case, both sequences $C_n(\omega)$, $D_n(\omega)$ diverge in this region as $n \to \infty$.

6. If in addition to the hypotheses in (1.) $\sum_{j=1}^{\infty} j\left(\left|a_j - 1/4\right| + \left|b_j\right|\right) < \infty$ holds, then

$$\lim_{n\to\infty} \frac{1}{n}\left[\lim_{\omega\to\pm 1} \frac{C_n(\omega)}{(1-\omega)}\right] = C(\pm 1)$$

$$\lim_{n\to\infty} \frac{1}{n}\left[\lim_{\omega\to\pm 1} \frac{D_n(\omega)}{(1-\omega)}\right] = D(\pm 1).$$

CorollaryForMeromorphicExtensionOfJFractions3

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z + b_1 - \cfrac{a_2}{z + b_2 - \cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$, and

$$\sum_{j=1}^{\infty} \left(\left| \frac{a_j - 1}{4} \right| + \left| b_j \right| \right) R^j < \infty$$

for some $R > 1$. Furthermore, let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$ and for notational convenience, let $u_j = 2\, b_j$ for $j \geq 0$ and let $v_j = 1 - 4\, a_j$ for $j \geq 1$. Let C, D be functions defined such that $C(\omega) = S_0(\omega)$, $D = S_{-1}(\omega)$ for

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k < j_1 < j_2 < \cdots < j_r < n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1 - w)^{-1} \left(\omega\, u_j \left(1 - w^{j-k}\right) + w\, v_j \left(1 - w^{j-k-1}\right)\right),$$

with $c_{k,j}(\pm 1) = \pm (j - k)\, u_j + (j - k - 1)\, v_j$ by definition. Then both C and D are holomorphic for $|\omega| < R^{1/2}$, both are continuous for $|\omega| \leq R^{1/2}$, and together they satisfy the identity

$$\omega^{-1}\, C\left(\omega^{-1}\right) D(\omega) - \omega\, C(\omega)\, D\left(\omega^{-1}\right) = \left(\omega^{-1} - \omega\right) \prod_{j=1}^{\infty} \left(1 - v_j\right)$$

for $R^{-1/2} \leq |\omega\,`| \leq R^{1/2}$.

CorollaryForMeromorphicExtensionOfJFractions4

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z+b_1-\cfrac{a_2}{z+b_2-\cfrac{a_3}{\ddots}}}}$$

where a_n, $b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$ hold. Furthermore, let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$ and for notational convenience, let $u_j = 2\, b_j$ for $j \geq 0$ and let $v_j = 1 - 4\, a_j$ for $j \geq 1$. Moreover, suppose the functions $C(\omega)$, $D(\omega)$ are defined to be $C(\omega) = S_0(\omega)$, $D = S_{-1}(\omega)$ for

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\, u_j\left(1-w^{j-k}\right) + w\, v_j\left(1-w^{j-k-1}\right)\right),$$

with $c_{k,j}(\pm 1) = \pm (j-k)\, u_j + (j-k-1)\, v_j$ by definition. Finally, define the functions A^+, A^-, B^+, B^- as follows: $A^+(x) = 2\, e^{-i\vartheta} C\left(e^{-i\vartheta}\right)$, $A^-(x) = 2\, e^{i\vartheta} C\left(e^{i\vartheta}\right)$, $B^+(x) = D\left(e^{-i\vartheta}\right)$, and $B^-(x) = D\left(e^{i\vartheta}\right)$. Given this framework, $-1 \leq x \leq 1$ implies that

$$2\pi\phi(x) = f^-(x) - f^+(x)$$

and that

$$\pi\, i\, \phi(\cos(\vartheta)) = e^{i\vartheta} C\left(e^{i\vartheta}\right) / D\left(e^{i\vartheta}\right) - e^{-i\vartheta} C\left(e^{-i\vartheta}\right) / D\left(e^{-i\vartheta}\right),$$

where

$$\phi(x) = \frac{2}{\pi}\left(1-x^2\right)^{1/2} \prod_{j=1}^{\infty}\left(1-v_j\right) / B^+(x)\, B^-(x)$$

for $x \in [-1, 1]$ with all roots nonnegative, where $x = \cos(\vartheta)$, $\vartheta \in [0, \pi]$ implies

$$\phi(\cos(\vartheta)) = \frac{2}{\pi} \sin(\vartheta) \prod_{j=1}^{\infty}\left(1-v_j\right) / D\left(e^{i\vartheta}\right) D\left(e^{-i\vartheta}\right),$$

and where $f^{\pm}(x)$ satisfy $f^{\pm}(x) = A^{\pm}(x) / B^{\pm}(x)$ for $-1 \leq x \leq 1$.

CReducedIrrationalNumber

In irrational number $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ with conjugate α' is C-reduced if $\alpha > 0$ and $\alpha' < -1$.

CRegularFractionsConvergeToIrrationals

Any C-regular continued fraction ξ converges to some $\alpha \in \mathbb{R} \backslash \mathbb{Q}$.

Criterion:ContinuedFractionTranscendence1

Let ξ be a positive irrational number $0 < \xi < 1$ with continued fraction expansion

$$\xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

with $a_j \in \mathbb{Z}^+$ and convergents A_n / B_n (with $q_{-1} = 0$). If the sequence $\{b_n\}_{n=1}^{\infty}$

1. is not eventually periodic

2. there exists a sequence of finite word $\{V_n\}_{n=1}^{\infty}$ such that V_n^w is a prefix of $\{b_n\}_{n=1}^{\infty}$

3. the sequence $\{|V_n|\}_{n=1}^{\infty}$ is increasing

and either there exists a rational $w \geq 2$, or there exists a rational $w > 1$ and the sequence $\left\{B_n^{1/n}\right\}_{n=1}^{\infty}$ is bounded, then ξ is transcendental.

Here, $|V_n|$ denotes the length of a word and V_n^w is the word formed by $\lfloor w \rfloor$ copies of V_n concatenated with the first $\lceil (w - \lfloor w \rfloor) \, |w| \rceil$ elements of V_n.

Criterion:ContinuedFractionTranscendence2

Let ξ be a positive irrational number $0 < \xi < 1$ with continued fraction expansion

$$\xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

with $a_j \in \mathbb{Z}^+$ and convergents A_n / B_n (with $q_{-1} = 0$). If the sequence $\{B_n\}_{n=1}^{\infty}$ is bounded define

$$m = \lim_{n\to\infty} \inf B_n^{1/n}$$

$$M = \lim_{n\to\infty} \sup B_n^{1/n}$$

and let two rational numbers $w > 1$ and v be chosen so that

$$w > (2\,v + 1)\,\frac{\ln(M)}{\ln(m)} - v.$$

If there exist two sequences $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$ such that

1. for any $n \geq 1$ the $U_n\, V_n^w$ is a prefix of $\{b_n\}_{n=1}^{\infty}$

2. the sequence $\{|U_n| / |V_n|\}_{n=1}^{\infty}$ is bounded from above by v

3. the sequence $\{|V_n|\}_{n=1}^{\infty}$ is increasing

then ξ is transcendental.

Here, $|V_n|$ denotes the length of a word and V_n^w is the word formed by $\lfloor w \rfloor$ copies of V_n concatenated with the first $\lceil (w - \lfloor w \rfloor) \, |w| \rceil$ elements of V_n.

CriterionForCConvergents

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational. Any ratio $A/B \in \mathbb{Q}$ satisfying

$$\left|\alpha - \frac{A}{B}\right| < \frac{1}{c_j\, q^2}, \quad j \in \{0, 1\}$$

where $c_0 = 3/2$ and $c_1 = 2$ is a C-convergent of α.

CriterionForCDualConvergents

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational. Any ratio $A/B \in \mathbb{Q}$ satisfying

$$\left|\alpha - \frac{A}{B}\right| < \frac{1}{c_j^*\, q^2}, \quad j \in \{0, 1\}$$

where $c_0^* = 2$ and $c_1^* = 3/2$ is a C-dual convergent of α.

CriterionForConvergenceOfGrommerFractions1

If it is possible to find a single bounded, nondecreasing function $\zeta(t)$ such that

$$\int_{-\infty}^{\infty} t^s\, d\zeta(t) = c_s$$

for $s = 0, 1, \ldots$, where $\zeta(-\infty) = 0$ by definition, then the associated continued fraction ξ of the form

$$\xi = \cfrac{c_0}{z - \alpha_0 - \cfrac{\beta_0}{z - \alpha_1 - \cfrac{\beta_1}{z - \alpha_2 - \cdots - \cfrac{\beta_{r-2}}{z - \alpha_{r-1} - \cdots}}}}$$

for a given formal power series $f_0(z) = \sum_{n=0}^{\infty} c_n\, z^{-n-1}$ is a Grommer fraction which converges to the value

$$\int_{-\infty}^{\infty} \frac{d\zeta(t)}{z - t},$$

$\operatorname{Im}(z) > 0$, of by the Stieltjes transform.

CriterionForConvergenceOfGrommerFractions2

Given a function f_0 which behaves asymptotically as the formal power series $P(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$ in the sector $\epsilon \leq \arg(z) \leq \pi - \epsilon$, $0 < \epsilon < \pi/2$, where $c_n \in \mathbb{R}$ for all n, where $f_0(z)$ is analytic for all $\text{Im}(z) > 0$, and where $\text{Im}(f_0(z)) < 0$ when $\text{Im}(z) > 0$, then the Grommer fraction ξ associated to f_0, P, converges whenever

$$\liminf_{n\to\infty} \left(\frac{c_{2n}}{(2n)!}\right)^{1/n} < \infty.$$

CriterionForExistenceOfGrommerFractionsForCertainPowerSeries1

The associated continued fraction ξ of the form

$$\xi = \frac{c_0}{z - \alpha_0 - \frac{\beta_0}{z - \alpha_1 - \frac{\beta_1}{z - \alpha_2 - \cdots - \frac{\beta_{r-2}}{z - \alpha_{r-1} - \cdots}}}}$$

for a given formal power series $f_0(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$ is a Grommer fraction if $H_k > 0$ for all $k = 0, 1, 2, \ldots$, where H_k denotes the kth Hankel determinant of f_0.

CriterionForExistenceOfGrommerFractionsForCertainPowerSeries2

The associated continued fraction ξ of the form

$$\xi = \frac{c_0}{z - \alpha_0 - \frac{\beta_0}{z - \alpha_1 - \frac{\beta_1}{z - \alpha_2 - \cdots - \frac{\beta_{r-2}}{z - \alpha_{r-1} - \cdots}}}}$$

for a given formal power series $f_0(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$ is a Grommer fraction if it is possible to find a bounded, nondecreasing function $\zeta(t)$ such that, for $s = 0, 1, \ldots$,

$$\int_{-\infty}^{\infty} t^s \, d\zeta(t) = c_s$$

where $\zeta(-\infty) = 0$ by definition.

CriterionForPowerSeriesToHaveSFractionExpansions

A power series of the form

$$\frac{c_0}{z} + \frac{c_1}{z^2} + \frac{c_2}{z^3} + \cdots$$

has an S-fraction expansion if and only if the determinants Δ_p and Ω_p are nonzero for all $p = 0, 1, 2, \ldots$ where for each p,

$$\Delta_p = \begin{vmatrix} c_0 & c_1 & \cdots & c_p \\ c_1 & c_2 & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{2p} \end{vmatrix}$$

and

$$\Omega_p = \begin{vmatrix} c_1 & c_2 & \cdots & c_{p+1} \\ c_2 & c_3 & \cdots & c_{p+2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p+1} & c_{p+2} & \cdots & c_{2p+1} \end{vmatrix}.$$

Criterion:SeidelSternCriterion

Let

$$\xi = b_0 + \mathop{\mathrm{K}}_{k=1}^{N} \frac{1}{b_k}$$

be a positive continued fraction (meaning $b_n \geq 0$ for all n). Then the continued fraction ξ converges if and only if

$$\sum_{n=1}^{m} b_n = \infty.$$

Criterion:TietzeCriterion

The continued fraction

$$\xi = b_0 + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k}{b_k}$$

with $a_k \in \mathbb{Z}$, $a_j \neq 0$ for all j and $b_k \in \mathbb{Z}^+$ converges if there exists a positive integer N such that for all $k \geq N$

$b_k \geq |a_k|$

$b_k \geq |a_k| + 1$ for $a_{k+1} < 0$.

Furthermore, if the continued fraction converges, the limit ξ is irrational.

DajaniKraaikampTwoDimensionalGaussKuzminTheorem

Let T be a Gauss map, $U : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$U(x,\ y) = \left\{T(x),\ \frac{1}{\left\lfloor \frac{1}{x} \right\rfloor + y}\right\},$$

λ be the Lebesgue measure on $\mathbb{R}^2$, $J(x,\ y) = (0,\ x) \times (0,\ y)$, $m_N(x,\ y)$ be given by

$$m_N(x,\ y) = \lambda\left(\left(U^N\right)^{-1}(J(x,\ y))\right),$$

and

$$g = \phi^{-1}.$$

Then

$$m_N(x,\ y) = \frac{\ln(1 + x\, y)}{\ln(2)} + O\left(g^N\right).$$

DarmonMckayContinuedFractionForOneOverEMinus1

Let ξ be a regular continued fraction where

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{n}{n}.$$

Then

$$\xi = \frac{1}{e - 1}.$$

DavisonFractions

Let θ be a positive irrational number $0 < \xi < 1$ and let $k \in \mathbb{Z}^+$ and $k \geq 2$. Then the continued fractions

$$\xi_k = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{1 + ((j\, \theta) \bmod k)}$$

are transcendental.

DavisonShallitSelfSimilarContinuedFractionsAreTranscendental

Let w_n be a sequence of natural numbers defining

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

a regular continued fraction, with convergents A_n / B_n that satisfy

$b_0 = 0$

$b_1 = w_0$

$\forall\, n \geq 0,\ b_{n+2} = B_n\, w_{n+1}.$

Then ξ is a transcendental number.

DawsonConvergenceCriterionI

Let ξ be a regular continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents g_n and suppose that g_{2n+1} converges absolutely and that g_{2n} converges, then g_n converges if and only if

$$\sum_{i=1}^{\infty} |b_{2i-1}| = \infty \bigvee \limsup \sum_{i=1}^{p} |b_{2i}| = \infty.$$

DawsonConvergenceCriterionII

Let ξ be a regular continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k}$$

with convergents $f_n = A_n / B_n$. If f_{2n} converges and $\exists\, k > 0,\ \forall_{i>k}\, B_i \neq 0$ and $\liminf(|a_n|) < \infty$, then there exists v and a subsequence q_n such that

$\lim f_{2\,q(n)+1} = v \bigwedge \lim f_{2n} = v.$

DawsonConvergenceCriterionIII

Let ξ be a generalized continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{1}$$

and r_n is a sequence of nonegative reals such that $r_1 |a_1 + 1| \geq |a_1|$, $r_2 |a_1 + a_2 + 1| \geq |a_2|$, for all $n \geq 3$, $r_n |a_{n-1} + a_n + 1| \geq r_{n-2}\, r_n |a_{n-1}| + |a_n|$ and

$$\liminf_n \prod_{i=1}^{n} r_i = 0,$$

and for all $n \geq 1$, $r_n < 1$

$$\sum_{i=1}^{\infty} (1 - r_i) = \infty.$$

Then ξ converges in the wider sense.

DawsonConvergenceCriterionIV

Let ξ be a generalized continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{1}.$$

Then if for all $n \geq 1$, $|a_n + a_{n+1} + 1| \geq 2 \max(|a_n|, |a_{n+1}|)$, ξ converges.

DegertConditionPeriods

Let d be a squarefree integer,

$$d = r + X^2$$

and

$$x = \sqrt{d}$$

be quadratic irrational numbers, ξ be the regular continued fraction of x, and l be the regular continued fraction period of ξ. Given $(4\, X) \bmod r = 0$ and $2 - 2\, X \leq r \leq 2\, X$, then $l \leq 12$.

Denseness Of Error Sum Functions Of Continued Fractions

Let α_n be an irrational number where $0 \leq \alpha_n \leq 1$, ξ_n be the regular continued fraction of α_n, $\mathcal{E}(\alpha_n)$ be the absolute error sum of ξ_n, $\mathcal{E}^*(\alpha_n)$ be the error sum of ξ_n, $S = [0, \phi]$, and $T = [0, 1]$. Then given that α_n is dense, it follows that $\mathcal{E}(\alpha_n)$ and $\mathcal{E}^*(\alpha_n)$ are dense in $[0, \phi]$ and $[0, 1]$, respectively.

DiscrepancyOfARealSequence

Let $E \subset [0, 1, \omega = \{x_n\}_{n=1}^{N}$ a sequence of real numbers and define $A(E; N; \omega)$ so that

$A\,(E; N; \omega) = \sharp\,\{n\ :\ 1 \leq n \leq N \text{ and } \operatorname{frac}(x_n) \in E\}$,

where $\sharp\,A$ denotes the number of elements of A for all sets A and $\operatorname{frac}(y)$ denotes the fractional part of the element y for all y.

The the discrepancy D_N associated with the finite segments of ω is defined to be

$$D_N\,(\omega) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{A([\alpha, \beta); N; \omega)}{N} - (\beta - \alpha) \right|.$$

DistributionalLimitForContinuedFractionsWithMinimalRemainder

Let x be a rational number where $0 \leq x \leq 1$,

$$\xi = \underset{n=1}{\overset{N}{\mathrm{K}}} \frac{a_n}{b_n}$$

be a half-regular continued fraction of x, define

$$S(x) = \sum_{n=1}^{N} a_n,$$

let M_n be rational numbers $0 \leq x \leq 1$ where $S(x) \leq n + 1$,

$$F_n(t) = \frac{\text{card}\,\{\xi : \xi \in M_n \bigwedge \xi \leq t\}}{\text{card}\,\{\xi : \xi \in M_n\}}$$

$$F(t) = \lim_{n \to \infty} F_n(t)$$

$$E(i) = \prod_{j=1}^{i} (-a_j)$$

$$A_i = \sum_{j=1}^{i} b_j$$

$$\lambda = \frac{1}{3} \left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right)$$

and

$$c = \frac{1}{\lambda - 1}.$$

Then

$$F(x) = b_0 - c\,\lambda \sum_{i=1}^{\infty} \frac{E(i)}{\lambda^{A_i}}.$$

DistributionalLimitForContinuedFractionsWithOddPartialQuotients

Let x be a rational number where $0 \leq x \leq 1$,

$$\xi = \underset{n=1}{\overset{N}{\mathrm{K}}} \frac{a_n}{b_n}$$

be the continued fraction with odd partial quotients continued fraction of x,

$$S(x) = \sum_{n=1}^{N} a_n,$$

M_n be rational numbers $0 \leq x \leq 1$ where $S(x) \leq n+1$,

$$F_n(t) = \frac{\operatorname{card}\{\xi : \xi \in M_n \bigwedge \xi \leq t\}}{\operatorname{card}\{\xi : \xi \in M_n\}}$$

$$F(t) = \lim_{n \to \infty} F_n(t)$$

$$E(i) = \prod_{j=1}^{i} (-a_j)$$

$$A_i = \sum_{j=1}^{i} b_j - 1$$

$$\lambda = \frac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right).$$

Then

$$F(x) = 1 - \sum_{i=1}^{\infty} \frac{E(i)}{\lambda^{A_i}}.$$

DistributionForMaximumPartialQuotient

Let α be an irrational number where $0 \leq \alpha \leq 1$,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be its regular continued fraction,

$$L_N = \max_{n \leq N} b_n,$$

y be a positive real, $S(N, y)$ be irrational numbers α where $0 \leq \alpha \leq 1$ and $L_N / N < y / \ln(2)$, and μ be the Gauss measure. Then

$$\lim_{N \to \infty} \mu(S(N, y)) = e^{-1/y}.$$

DistributionOfRationalsWRTLargestPartialDenominator

Let $0 < p/q < 1$ be a rational number and $\gcd(p, q) = 1$. Let $D(p/q)$ be the maximal partial denominator that occurs in the regular continued fraction expansion of p/q

$$D\left(\frac{p}{q}\right) = \max_{\frac{p}{q} = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k} \bigwedge b_N > 1} (\{b_1, b_2, \ldots, b_N\}),$$

and let $\Phi(x, \alpha)$ be the number of fractions with maximal denominator x such that their largest partial denominator is less than $\ln(\alpha)\, x$

$$\Phi(x, \alpha) = \mathrm{card}\left(\frac{p}{q : 0 \le p < q \le x \bigwedge \gcd(p, q) = 1 \bigwedge D\left(\frac{p}{q}\right) < \alpha \ln(x)}\right).$$

Then for $\alpha > 4/\ln(\ln(x))$

$$\Phi(x, \alpha) = \frac{3}{\pi^2}\, x^2\, e^{-\frac{12}{\alpha \pi^2}} \left(1 + O\left(\frac{1}{\alpha^2} + 1\right) e^{\frac{24}{\alpha \pi^2}}\, \frac{\ln(\ln(x))}{\ln(x)}\right)$$

holds uniformly.

DistributionOfTheLargestPartialDenominator

Let $0 < p/q < 1$ be a rational number and $\gcd(p, q) = 1$. Let $D(p/q)$ be the maximal partial denominator that occurs in the regular continued fraction expansion of p/q

$$D\left(\frac{p}{q}\right) = \max_{\frac{p}{q} = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k} \bigwedge b_N > 1} \{b_1, b_2, \ldots, b_N\}.$$

and let $\Phi(x, \alpha, M)$ be the number of fractions with maximal denominator x such that exactly M of their partial denominator are greater than $\alpha \ln(x)$

$$\Phi(x, \alpha, M) = \mathrm{card}\left(p \middle/ \left(q : 0 \le p < q \le x \bigwedge \gcd(p, q) = 1 \bigwedge \mathrm{card}\left(\left\{ b_{j : b_j > \alpha \ln(x) \bigwedge \frac{p}{q} = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k} \bigwedge b_N > 1} \right\}\right) = M\right)\right).$$

Then asymptotically for large x

$$\Phi(x, \alpha, M) = \frac{3}{\pi^2}\, x^2\, e^{-12/(\alpha \pi^2)}\, \frac{1}{M!} \left(\frac{12}{\alpha \pi^2}\right)^M.$$

DomainOfConvergenceAssociatedToRogersRamanujanCon

tinuedFraction

Let τ be an irrational number, define the modular nome by

$q = e^{2 i \pi \tau}$,

let $\xi(q)$ be the Rogers Ramanujan continued fraction of q,

$$G_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q;q)_m}$$

be a holomorphic function, R_q be the holomorphic radius of $G_q(z)$,

$$H_q(z) = \frac{G_q(z)}{G_q(q\,z)}$$

be a meromorphic function, V_q be the poles of $H_q(z)$ in D, a complex disk with radius R_q, Ω_q be circles containing the poles in V_q,

$U = D - \Omega_q$

be a complex domain, and X be any closed set where $X \subset \Omega_q$, $X \neq \Omega_q$.

Then $\xi(q)$ converges uniformly to $H_q(z)$ on compact sets in U and for all X it is not true that $\xi(q)$ converges uniformly on compact sets $D - X$.

DomainOfConvergenceForRogersRamanujanContinuedFraction

Let τ be an irrational number, define the modular nome by

$q = e^{2 i \pi \tau}$,

$\xi(q)$ be the Rogers Ramanujan continued fraction of q,

$$G_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q;q)_m}$$

be a holomorphic function,

$$H_q(z) = \frac{G_q(z)}{G_q(q\,z)}$$

be a meromorphic function, V_q be the poles from $H_q(z)$ in D, the unit disk, Ω_q be complex circles containing the points in V_q,

$U = D - \Omega_q$

be a complex domain, and X be any closed set where $X \subset \Omega_q$, $X \neq \Omega_q$.

Then $\xi\,(q)$ converges uniformly to $H_q(z)$ on compact sets in U and

$(\forall_X$ it is not true that $\xi(q)$ converges uniformly on compact sets in $D - X)$.

DuallyRegularChain

A dually regular chain is an infinite product $T_0\, T_1 \cdots T_n \cdots$ where $T_0 = V_1^{b_0}$, $b_0 \in \mathbb{Z}$, $T_1 \neq V_1$, and

$$\begin{cases} T_n \in \{V_j,\, C\} & \text{for } \det(T_0\, T_1 \cdots T_{n-1}) = \pm 1 \\ T_n \in \{V_j,\, E_j,\, C\} & \text{for } \det(T_0\, T_1 \cdots T_{n-1}) = \pm i \end{cases}$$

for $n \geq 1$ such that no $n_0 \in \mathbb{Z}^+$, $j \in \{1, 2, 3\}$ exist for which $T_n = V_j$ for all $n \geq n_0$.

The matrices used here are defined as follows:

$$V_1 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix},\ V_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix},\ V_3 = \begin{pmatrix} 1-i & i \\ -i & i+1 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & i-1 \\ 0 & i \end{pmatrix},\ E_3 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & i-1 \\ 1-i & i \end{pmatrix}.$$

EigenvaluesOfGaussKuzminWirsingOperator

Let $\mathcal{L}$ be the Gauss Kuzmin Wirsing operator and λ_n be its eigenvalues. Then $|\lambda_{1+n}| < |\lambda_n|$, λ_n has simple eigenvalues, $(-1)^{1+n}\, \lambda_n > 0$, and

$$\lim_{n \to \infty} \frac{\lambda_n}{\lambda_{1+n}} = \frac{1}{2}\left(-3 - \sqrt{5}\right).$$

EquivalenceTransformation

Two continued fractions ξ and ξ' of the forms

$$\xi = b_0 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{a_m}{b_m}$$

and

$$\xi' = b'_0 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{a'_m}{b'_m}$$

are said to be equivalent if there exists a sequence of complex numbers $r = \{r_m\}$ with $r_0 = 1$, $r_m \neq 0$ for $m \geq 1$, so that $b'_0 = b_0$, $a'_m = r_m\, r_{m-1}\, a_m$, and $b'_m = r_m\, b_m$ for $m = 1, 2, 3, \ldots$. Here, the sequence r is said to be an equivalence transformation between ξ and ξ'.

Perhaps the most commonly-used equivalence transformations results when r_m has the form

$$r_m = \prod_{k=1}^{m} a_k^{(-1)^{m+1-k}} = \left(\frac{\prod_{k=1}^{\lfloor m/2 \rfloor} a_{2\,k}}{\prod_{k=1}^{\lfloor (m+1)/2 \rfloor} a_{2\,k-1}} \right),$$

which transforms ξ into its regular continued fraction form ξ_{reg}. Here, ξ_{reg} is a regular continued fraction of the form

$$\xi_{\text{reg}} = b_0 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{d_m},$$

where $d_1 = b_1 / a_1$, and for $m = 1, 2, 3, \ldots$,

$$d_{2\,m} = b_{2\,m} \frac{a_1\, a_3 \cdots a_{2\,m-1}}{a_2\, a_4 \cdots a_{2\,m}},$$

$$d_{2\,m+1} = b_{2\,m+1} \frac{a_2\, a_4 \cdots a_{2\,m}}{a_1\, a_3 \cdots a_{2\,m-1}}.$$

EquivalenceTransformationNumeratorDenominatorCancellation

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{\frac{\alpha_k}{\beta_k}}{\frac{\gamma_k}{\delta_k}}$$

be a continued fraction with convergents A_k / B_k. Then the continued fraction

$$\eta = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \begin{cases} \frac{\alpha_1 \delta_1}{\beta_1 \gamma_1} & \text{for } k = 1 \\ \frac{\alpha_k \delta_k \beta_{k-1} \gamma_{k-1}}{\beta_k \gamma_k} & \text{for } k > 1 \end{cases}$$

with convergents P_k / Q_k is equivalent to the continued fraction ξ, meaning

$\eta = \xi$

$P_k = A_k$

$Q_k = B_k$.

EquivalenceTransformationWithUnitDenominator

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

with $b_k \neq 0$ for $k \geq 1$ be a continued fraction with convergents A_k / B_k. Then the continued fraction

$$\eta = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{\begin{cases} \frac{a_1}{b_1} & \text{for } k = 1 \\ \frac{a_k}{b_{k-1} b_k} & \text{for } k > 1 \end{cases}}{1}$$

with convergents P_k / Q_k is equivalent to the continued fraction ξ, meaning

$\eta = \xi$

$P_k = A_k$

$Q_k = B_k$.

EquivalenceTransformationWithUnitNumerator

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

with $a_k \neq 0$ for $k \geq 1$ be a continued fraction with convergents A_k / B_k. Then the continued fraction

$$\eta = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{\begin{cases} \dfrac{\prod_{j=1}^{k/2} a_{2j-1}}{\prod_{j=1}^{k/2} a_{2j}} \times b_k & \text{for } k \text{ even} \\ \dfrac{\prod_{j=1}^{(k-1)/2} a_{2j}}{\prod_{j=1}^{(k+1)/2} a_{2j-1}} \times b_k & \text{for } k \text{ odd} \end{cases}}$$

with convergents P_k / Q_k is equivalent to the continued fraction ξ, meaning

$\eta = \xi$

$P_k = A_k$

$Q_k = B_k$.

EquivalentSternStolzSeriesDivergenceCriteria

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction. The Stern-Stolz series of ξ diverges if one of the following three criteria holds:

1. $\lim_{m \to \infty} \left(\sum_{n=2}^{m} \sqrt{\left| \frac{b_{n-1}\, b_n}{a_n} \right|} \right) = \infty$

2. $\lim_{m \to \infty} \left(\sum_{n=2}^{m} \sqrt{\left| \frac{b_{n-1}\, b(n)}{n\, a_n} \right|} \right) = \infty$

3. $\lim_{m \to \infty} \inf_{2<n<m} \left(\frac{a_n}{b_{n-1}\, b_n} \right) < \infty$.

EstimatesForHausdorffDimensionForConstrainedPartialQuotients

Let E be a subset of the natural numbers, $E(R)$ be the regular continued fractions ξ whose partial denominators lie in E, $E(G)$ be the backwards continued fractions ξ whose partial denominators lie in E, and H be the Hausdorff dimension. Then given

$$\sum_{e \in E} \frac{1}{e} = \infty$$

it follows that $H(E(R)) \geq 1/2$ and $H(E(G)) \geq 1/2$.

EstimatingIntegralsUsingAlgebraicIrrationals

For an arbitrary function f of bounded variation, denote by I the integral $I = \int_0^1 f(x)\,dx$. If α is any algebraic irrational in $(0, 1)$ whose continued fraction $\xi = [0; b_1, b_2, \ldots]$, then $I_N - I = O\left(N^{-(1-\epsilon)}\right)$ where $\epsilon > 0$ and where

$$I_N = \frac{1}{N} \sum_{k=1}^{N} f(k\,\alpha).$$

EstimatingIntegralsUsingQuadraticIrrationalContinuedFractions

For an arbitrary function f of bounded variation, denote by I the integral $I = \int_0^1 f(x)\,dx$. If α is an irrational in $(0, 1)$ whose continued fraction $\xi = [0; b_1, b_2, \ldots]$ has bounded partial denominators, then $I_N - I = O(\ln(N)/N)$ where

$$I_N = \frac{1}{N} \sum_{k=1}^{N} f(\mathrm{k}\alpha).$$

Moreover, if f is a characteristic function of some subinterval J of $(0, 1)$, then

$$|I_N - I| \leq 36 \cdot \sup_k \{b_k\} \cdot \frac{\ln(N)}{N}.$$

EstimatingIntegralsUsingSlowlyConvergingContinuedFractions

For an arbitrary function f of bounded variation, denote by I the integral $I = \int_0^1 f(x)\, dx$. If α is an irrational in $(0, 1)$ whose continued fraction $\xi = [0; b_1, b_2, \ldots]$ has rational approximants of the form A_k / B_k where $B_0 = 1$ and where $B_{j+1} = b_{j+1} B_j + b_{j-1}$ for $j = 1, 2, \ldots$, and if each partial denominator B_j of ξ satisfies the relation $B_{j+1} = O\left(B_j^\gamma\right)$ for fixed $\gamma > 1$, then $I_N - I = O\left(N^{-1/\gamma}\right)$ where

$$I_N = \frac{1}{N} \sum_{k=1}^{N} f(k\,\alpha).$$

EstimationOfApproximantsForLimitPeriodicContinuedFractions

Let $\xi = \mathrm{K}(b_n / 1) = [0; b_1, b_2, \ldots]$ be a limit periodic continued fraction which satisfies for all $n = 1, 2, \ldots$ $d_n \le \frac{1}{4\left(4 n^2 - 1\right)}$, where $d_n = \max_{m \ge n} |a_m - (-1/4)|$. In particular, if $S_n(0) = A_n / B_n$ is the nth approximant of ξ and if the approximant function $S_n(w) = \frac{A_n + A_{n-1} w}{B_n + B_{n-1} w}$ for all complex w, the following estimates are valid:

$$\left| \frac{\xi - S_n\left(-\frac{1}{2}\right)}{\xi - S_n(0)} \right| \le \begin{cases} \frac{1-\beta}{1+\beta}\left(1 + \frac{\beta}{n} + \frac{2\beta+1}{2 n^2}\right) & \text{for } d_n \le \frac{1-\beta^2}{4\left(4 n^2 - 1\right)},\ 0 \le \beta \le 1,\ n \ge 1 \\ \frac{4 d (n+1)(n+2)}{(n+1)^{d+1} - 2 d} & \text{for } d_n \le \frac{d}{2 n^{\alpha+1}},\ \alpha > 1,\ d > 0 \\ \frac{4 (n+1) r^{n+1} (2+r)}{(1-r)\left(1 - 4 r^{n+1}\right)} & \text{for } d_n \le r^n,\ 0 < r < 1. \end{cases}$$

For the second case, the estimate holds for $(n-1)^\alpha (\alpha - 1) > 2\, d\, n$ and for the third, the estimate holds whenever $(1-r)^2 > 18\, r^{n+1}$.

EulerMindingFormulas

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction and A_k / B_k the sequence of its convergents.

Then the following explicit forms for the numerators and denominators of the convergents hold:

$$A_n = \left(\prod_{i=0}^{n} b_i\right) \times \left(1 + \sum_{\mu=0}^{n-1} \left(\sum_{i_\mu=0}^{\mu-1} \sum_{i_{\mu-1}=0}^{i_\mu-1} \cdots \sum_{i_1=0}^{i_2-1} \sum_{i_0=0}^{i_1-1} \prod_{m=0}^{\mu} \frac{a_{i_m+m+1}}{b_{i_m+m}\, b_{i_m+m+1}}\right)\right)$$

$$B_n = \left(\prod_{i=1}^{n} b_i\right) \times \left(1 + \sum_{\mu=0}^{n-1} \left(\sum_{i_\mu=1}^{\mu-1} \sum_{i_{\mu-1}=1}^{i_\mu-1} \cdots \sum_{i_1=1}^{i_2-1} \sum_{i_0=1}^{i_1-1} \prod_{m=0}^{\mu} \frac{a_{i_m+m+1}}{b_{i_m+m}\, b_{i_m+m+1}}\right)\right).$$

EulerQuadraticIrrationalTheorem

Let

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

be a continued fraction with $b_n \in \mathbb{Z}^+$ and $b_n > 0$ for $n > 0$ and $b_{n+j} = b_n$ for all $n \geq n_0 \geq 0$ for some $j \geq 0$. Then ξ is a quadratic irrational, meaning ξ is the solution of a quadratic equation with rational coefficients.

EulersFirstContinuantIdentity

Let ξ be a regular continued fraction and $K(i,\ j)$ its classical continuant. Then for all $i < m < n < j$,

$K(i,\ j)\, K(m,\ n) - K(i,\ n)\, K(m,\ j) = (-1)^{j-m}\, K(i,\ m-2)\, K(j+2,\ n)$.

EulerWallisRecursion

Let

$$\xi = b_0 + \mathop{\mathrm{K}}_{k=1}^{N} \frac{a_k}{b_k}$$

be a continued fraction and A_k / B_k the sequence of its convergents. Then the following recursion relations hold:

$$A_k = a_k A_{k-2} + b_k A_{k-1}$$
$$B_k = a_k B_{k-2} + b_k B_{k-1}$$

with the initial condition $A_{-1} = 1$, $A_0 = b_0$, $B_{-1} = 0$, $B_0 = 1$.

EvenContraction

Let $\xi = b_0 + \mathrm{K}(a_m / b_m)$ be a generalized continued fraction with nth approximant $\xi_n = A_n / B_n$. A continued fraction $\zeta = d_0 + \mathrm{K}(c_m / d_m)$ with nth approximant $\zeta_n = C_n / D_n$ is said to be an even contraction of ξ if and only if $\zeta_n = \xi_{2n}$ for $n = 0, 1, 2, \ldots$. Note that ξ has an even contraction if and only if $b_{2n} \neq 0$ for all positive integers n.

EveryNumberInUnitIntervalIsSumOfKRealNumbersWhoseContinuedFractionsHavePartialQuotientsLessThanOrEqualToK

Define $F_k = \{\alpha \in (0, 1/k)$ such that its partial quotients are less than or equal to $k\}$ then

$k F_k = [0, 1]$.

EveryQuadraticIrrationalHasPeriodicCDuallyRegularFractionExpansion

Every irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ which is quadratic over $\mathbb{Q}$ has a periodic C-dually regular continued fraction expansion.

EveryQuadraticIrrationalHasPeriodicCRegularFractionExpansion

Every irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ which is quadratic over $\mathbb{Q}$ has a periodic C-regular continued fraction expansion.

EveryRealNumberIsProductOfTwoF4RegularContinuedFractions

Every real number $x \geq 1$ can be represented as a product of two regular continued fractions $x = \xi_1 \xi_2$

$$\xi_j = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with $0 \leq b_k \leq 4$ for all k and $i = 1, 2$.

EveryRealNumberIsSumOfTwoF4RegularContinuedFractions

Let T be the interval $\left[\sqrt{2} - 1, 4\sqrt{2} - 4\right]$. Then every real number $x \in T$ can be represented as a sum of two regular continued fractions $x = \xi_1 + \xi_2$

$$\xi_j = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with $0 \leq b_k \leq 4$ for all k and $i = 1, 2$.

ExactGaussKuzminLevyTheorem

Let τ be the Gauss map

$$\tau : \mathbb{R} \to \mathbb{Z}$$

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Let μ be the Lebesgue measure. Then

$$\mu\left(x : \tau^k(x) < z\right) = \frac{\ln(1+z)}{\ln(2)} + \sum_{m=2}^{\infty} \lambda_n^k \Phi_n(z).$$

Here λ_n are the eigenvalues of the Gauss-Kuzmin-Wirsing operator $\mathcal{L}$ and $\Phi_n'(z)$ the eigenfunctions of $\mathcal{L}$. The eigenfunctions fulfill

$$\Phi_n(0) = \Phi_n(1) = 0$$

$$\sup_{\mathrm{Re}(z) \geq -1/2} \left|(z+1)\Phi_n'(z)\right| < \infty.$$

ExistenceForArbitraryRadiusOfConvergenceForGSeriesAssociatedToRogersRamanujanContinuedFraction

Let x be a real number where $0 \leq x \leq 1$, τ be an irrational number, define the modular nome by

$q = e^{2 i \pi \tau}$,

let $\xi(q)$ be the Rogers Ramanujan continued fraction of q,

$$G_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q; q)_m}$$

be a holomorphic function, and R_q be the holomorphic radius set of $G_q(z)$. Then

$\forall_x \exists_\tau R_q = x$.

ExistenceOfConstantCoefficientVectorFieldsThatAreNotGloballyAnalyticHypoelliptic

Let α be an irrational whose continued fraction a_n has convergent denominators B_n satisfying

$a_{n+1} > e^{K B_n} / B_n$

where $K > 0$. Then

$V = d / \mathrm{dx} - \alpha\, d / \mathrm{dy}$

is neither globally analytic hypoelliptic nor globally hypoelliptic.

ExistenceOfRichardsGoldbergFractionsForPositiveRealFunctions

If f_r is a positive real function for which neither $a_r\, f_r\,(z) - z\, f_r\,(a_r)$ nor $a_r\, f_r\,(a_r) - z\, f_r\,(z)$ vanish for arbitrary positive constants $a_r \in \mathbb{R}$, then the function f_{r+1} defined by the recursive relation

$$f_{r+1}\,(z) = \frac{a_r\, f_r\,(z) - z\, f_r\,(a_r)}{a_r\, f_r\,(a_r) - z\, f_r\,(z)}$$

is positive real and has an associated continued fraction ξ_{r+1} of the form

$$\xi_{r+1} = \frac{d_0}{z} + \frac{e_1 - z^2}{d_1\, z - \frac{e_2 - z^2}{d_2\, z - \cdots}}$$

for some complex constants $d_1,\, d_2,\, \ldots\, ,\, e_1,\, e_2,\, \ldots$. The continued fraction ξ_{r+1} is called the Richards-Goldberg continued fraction associated with f_{r+1}.

ExistenceTheoremForEntireFunctionWhoseDiagonalPadeApproximantsConvergeNowhere

Let f be an entire function and $f_n(z)$ be the Padé approximants diagonals at 0. Then

$$\exists_f\, \forall_{z \neq 0} \limsup_{n \to \infty} |f_n(z)| = \infty.$$

ExistenceTheoremForSingularitiesOutsideConvergenceDiskForPadeApproximantRows

Let f be a meromorphic function, $D(m)$ be the largest complex disk where f has less than or equal to m poles.

Let $T_{m,n}$ be the m th row Padé approximants, R_m be the radius of $D(m)$, a be an element of $\mathbb{C}\backslash 0$, μ be a positive integer where $2 \leq \mu \leq m$, $U(a)$ be the poles converging from $T_{m,n}$ at a, a_j be elements of $\mathbb{C}\backslash 0$ where $0 < |a_1|$ and $|a_j| \leq |a_{j+1}|$,

$\forall_{\mu \leq j \leq m} |a_j| = R,$

and $Q_{n,m}$ be the Padé approximant denominators. Then given

$$\exists_{N>0} \forall_{n>N} Q_{n,m} = \prod_{j=1}^{m} (z - \zeta(j, n))$$

$$\forall_{1 \leq j \leq m} \lim_{n \to \infty} \zeta(j, n) = a_j,$$

it follows that

$$\forall_{-1+\mu \leq j \leq m} R_m = R$$

$V_m = \{a(1), \ldots, a(-1+\mu)\}$ are the poles of f in $D(m)$

$\forall_{\mu \leq j \leq m} a_j$ are singular points for f.

ExpressionForInvariantProbabilityOfBernoulliRandomContinuedFractionWithParameterAlpha

Let Z_n be an independent identically distributed Bernoulli random variable, P its probability expectation, X_n a Markov chain defined by

$$X_n = 1/X_{n-1} + Z_n$$

$$P(Z_n = 0) = \alpha$$

$$P(Z_n = 1) = 1 - \alpha.$$

Then X_n converges to a singular probability π supported on the positive reals which has the distribution function $F(x)$, that can be described by writing x as a continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

and then

$$F(x) = \begin{cases} \sum_{i=0}^{\infty} \left(-\frac{1}{\alpha}\right)^i \left(\frac{\alpha}{\alpha+1}\right)^{\sum_{j=1}^{i+1} a_j} & \text{for } x \leq 1 \\ 1 - \frac{F\left(\frac{1}{x}\right)}{\alpha} & \text{for } x > 1. \end{cases}$$

FareyInterval

Given a Farey pair $a/b < c/d$, the interval $[a/b,\ c/d]$ is called a Farey interval.

FareyPair

A pair of nonnegative rational numbers $a/b < c/d$ is called a Farey pair if $b\,c - a\,d = 1$, i.e., if $c/d - a/b = 1/(b\,d)$.

FarinhaConvergenceCriterion

Let ξ be a generalized continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{1}$$

where the a_k are functions in a region D satisfying

$$\lim_{n\to\infty} a_n(z) = 0 \bigwedge a_n(z) \neq 0$$

and $|a_1| \leq \alpha \wedge |a_1 + 1| \geq |a_1| + \mu$ for some α and μ for all $n \geq 1$, $|a_n + a_{n+1} + 1| \geq 2\max(|a_n|,\ |a_{n+1}|)$. Then ξ converges and $|\xi(z)| < \min(3/2,\ (\alpha + \mu)/\mu \wedge 2)$.

FastContinuedFractionAlgorithmGivesUltraCloseApproximationsToIrrationals

For any irrational number α in $(0,\ 1)$, the fast continued fraction algorithm gives precisely the set of all ultra-close approximations to α.

FastKhinchinSpectrumOfContinuedFractions

Let α be an irrational number where $0 \leq \alpha \leq 1$,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be its regular continued fraction, ψ_n be a sequence where

$$\lim_{n\to\infty} \frac{\psi_n}{n} = \infty,$$

$E(\psi)$ be irrational numbers α where

$$0 \leq \alpha \leq 1 \bigwedge \lim_{n\to\infty} \frac{\sum_{j=1}^{n} \ln(b_j)}{\psi_n} = 1,$$

$$c = \limsup_{n\to\infty} \frac{\psi(n+1)}{\psi(n)},$$

and H be the Hausdorff dimension. Then given ψ_n is monotonic increasing it follows that

$$H(E(\psi)) = \frac{1}{1+c}.$$

FickenContinuedFractionCypher

Using the correspondence A → 2, B → 3, … , any text message can be encoded in the convergents of a regular continued fraction

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

with the association $b_k \to letter$.

FiniteAutomatonBoundForGeneratingContinuedFractionsOfAlgebraics

Let α be an algebraic number where $0 < \alpha < 1$, d be the algebraic degree set of α,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction of α, and b_n be the partial denominator of ξ. Then given $d \geq 3$, it is not the case that b_n is an automatic sequence.

FoldingLemma

Let ξ be the regular continued fraction expansion

$$\xi = a_0 + \underset{j=1}{\overset{M}{\mathrm{K}}} \frac{1}{b_j}$$

and convergents A_n / B_n. Then the folowing identity holds for all $n \in \mathbb{Z}^+$, $n \leq M$, $x \in \mathbb{C} \setminus 0$:

$$\frac{A_n}{B_n} + \frac{(-1)^n}{x\, B_n^2} = b_0 + \underset{j=1}{\overset{n}{\mathrm{K}}} \frac{1}{\begin{cases} b_j & \text{for } 1 \leq j \leq n \\ x - \frac{B_{n-1}}{B_n} & \text{for } j = n+1 \end{cases}} =$$

$$b_0 + \underset{j=1}{\overset{2n+1}{\mathrm{K}}} \frac{1}{\begin{cases} b_j & \text{for } 1 \leq j \leq n \\ x & \text{for } j = n+1 \\ -b_{2n+2-j} & \text{for } 1 \leq j \leq 2n+1. \end{cases}}$$

FractionalPartsOfIrrationalsUniformlyDistributedModOne

For $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $\mathrm{frac}(n\,\theta)$ is uniformly distributed modulo one for $n = 1, 2, \ldots$, where $\mathrm{frac}(y)$ denotes the fractional part of y.

FunctionOfGaussMapAverageForAlmostAllNumbers

Let ξ be an irrational number from the interval $(0, 1)$ and let τ be the Gauss map

$\tau : \mathbb{R} \to \mathbb{Z}$

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Then for any measuerable function f, and for almost all ξ, the following iden tity holds:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\tau^k(\xi)\right) = \frac{1}{\ln(2)} \int_0^1 \frac{f(x)}{x+1}\, dx.$$

FundamentalFormulas

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction and A_k / B_k the sequence of its convergents. Further, let $A_{\lambda,\nu}$ be the numerator of the convergents of the continued fraction

$$b_\lambda + \underset{k=\lambda+1}{\overset{\lambda+\nu}{\mathrm{K}}} \frac{a_k}{b_k}$$

with initial conditions $A_{0,\nu} = A_\nu$, $A_{\lambda,-1} = 1$, $A_{\lambda,0} = b_\lambda$ and let $B_{\lambda,\nu}$ be the numerator of the convergents of the continued fraction

$$b_{\lambda+1} + \underset{k=\lambda+2}{\overset{\lambda+\nu}{\mathrm{K}}} \frac{a_k}{b_k}$$

with initial conditions $B_{0,\nu} = B_\nu$, $B_{\lambda,-1} = 0$, $B_{\lambda,1} = b_{\lambda+1}$.

Then the following recursion relations hold:

$q_{\lambda+\nu-1} = a_\lambda \, B_{\lambda-2} \, B_{\lambda,\nu-1} + B_{\lambda-1} \, A_{\lambda,\nu-1}$

$A_{\lambda+\nu-1} = a_\lambda \, A_{\lambda-2} \, B_{\lambda,\nu-1} + A_{\lambda-1} \, A_{\lambda,\nu-1}$.

GaloisPeriodicRegularContinuedFraction

Let $\xi > 1$ be a quadratic irrational, meaning a nonrational solution of a quadratic equation with rational coefficients of the form

$$\xi = \frac{P + \sqrt{D}}{Q}$$

with P, Q, $D \in \mathbb{Z}$ with $P \geq 0$, $D > 0$, and $Q > 0$, and $Q \mid \left(D - P^2\right)$. If its conjugate

$$\eta = \frac{P - \sqrt{D}}{Q}$$

satisfies $-1 < \eta < 0$, then ξ has a purely periodic regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with $b_{n+j} = b_n$ for all $n \geq 0$.

GaussKuzminTheoremForOptimalContinuedFractions

Let K be a simply connected set, B be its boundary, β_i and γ_i be real numbers where $-g^2 \leq \beta_i < \gamma_i$, κ_i and τ_i be real numbers where $0 \leq \kappa_i < \tau_i \leq 1/2$, f_i be a continuous function that is monotonic on $[\beta_i, \gamma_i]$.

Also let l_i be a parametrized curve

$l_i = \{\{\eta, f_i(\eta)\} \mid \beta_i \leq \eta \leq \gamma_i\} \vee l_i = \{\{\beta_i, \eta\} \mid \kappa_i \leq \eta \leq \tau_i\}$,

where

$B = \bigcup l_i$.

Finally, let $D_n(K)$ be real numbers where $-\frac{1}{2} \leq x \bigwedge x < \frac{1}{2} \bigwedge \exists_y T_{\text{ocf}}^{n(y,0)} \in K$, λ be the Lebesgue measure, μ be the optimal continued fraction measure,

$$f_{\text{ocf}}(t, v) = \left\lfloor \frac{\left\lfloor \frac{1}{|t|} \right\rfloor + v \operatorname{sgn}(t)}{2\left(\left\lfloor \frac{1}{|t|} \right\rfloor + v \operatorname{sgn}(t)\right) + 1} + \frac{1}{|t|} \right\rfloor$$

be a function,

$$T_{\text{ocf}}(t, v) = \left\{|t| - f_{\text{ocf}}(t, v),\ \frac{1}{f_{\text{ocf}}(t, v) + v \operatorname{sgn}(t)}\right\},$$

and

$g = \phi^{-1}$.

Then

$\lambda(D_n(K)) = \mu(K) + O(g^n)$.

GaussKuzminWirsingConstant

Let $G(x)$ denote the Gauss map which is defined piecewise to be

$$G(x) = \begin{cases} x & \text{for } x = 0 \\ x - \lfloor x \rfloor & \text{for } x \neq 0. \end{cases}$$

From this, one can define the Gauss-Kuzmin operator (sometimes called the Gauss-Kuzmin-Wirsing operator) h to be the transfer operator of the Gauss map G having the form

$$h(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

or, alternatively, the form in which it acts on functions f, namely

$$[G\, f](x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2}\, f\left(\frac{1}{x+n}\right).$$

Though analytic forms of its eigenfunctions are unknown past the zeroth such function, numerical methods can be used to compute the eigenvalues of the Gauss-Kuzmin operator. The first eigenvalue λ_1 is, to fifty decimal places, equal to

$\lambda_1 = -.30366300289873265859744812190155623311087735225365\ldots$, and the constant λ defined to be the absolute value $\lambda = |\lambda_1|$ of this first eigenvalue is, by definition, the Gauss-Kuzmin-Wirsing constant and is intimately connected to the study of continued fractions.

The discovery of this constant was a result of an early problem of Gauss who, at the time, was interested in the probability distribution of coefficients in the continued fraction expansion of a random variable uniformly distributed in (0, 1). To that end, given an arbitrary number x uniformly distributed in (0, 1) with regular continued fraction expansion $\xi(x) = [0; b_1, b_2, \ldots, b_n, \ldots]$, Gauss was able to find for all $b \in \mathbb{Z}^+$ a closed-form asymptotic equivalence for the value $\Pr\{b_n = b\}$ as $n \to \infty$, namely

$$\lim_{n\to\infty} \Pr\{b_n = b\} = -\log_2\left(1 - \frac{1}{(b+1)^2}\right).$$

Moreover, it was proved that if $r_n = r_n(x) = [b_n; b_{n+1}, b_{n+2}, \ldots]$ and if $z_n(x) = r_n - b_n = [0; b_{n+1}, b_{n+2}, \ldots]$, then the (Lebesgue) measure $m_n(\alpha)$ of the collection of all numbers $x \in (0, 1)$ for which $z_n(x) < \alpha$ satisfies the asymptotic result

$$\lim_{n\to\infty} m_n(\alpha) = \frac{\ln(1+\alpha)}{\ln(2)},$$

$\alpha \in [0, 1]$. The goal then shifted to finding an expression for the value of the expression $m_n(x) - \ln(1+x)/\ln(2)$ for large values n, and no solution belonging to Gauss was ever published.

Later, Kuzmin published the first solution to this problem. He proved that by setting

$$m_n(x) = \frac{\ln(1+x)}{\ln(2)} + r_n(x),$$

the value of $r_n(x)$ satisfied the asymptotic result $r_n(x) = O\left(q^{\sqrt{n}}\right)$ for a constant $q \in (0, 1)$ independent of n, x. Later, Lévy was able to bound $r_n(x)$ asymptotically by 0.7^n and even later, Szüsz was able to improve the bound to 0.485^n. In the mid 1970s, Wirsing gave the exact asymptotic bounds for m_n to be

$$m_n(x) = \frac{\ln(1+x)}{\ln(2)} + (-\lambda)^n\,\Psi(x) + O(x(1-x)\,\mu^n)$$

for a specifically defined function Ψ and a unique constant μ while simultaneously computing the value λ accurately to ten decimal places.

Much work has been done to advance the computational accuracy and theoretical understanding of the constant λ since Wirsing's work was published. For example, mathematicians Babenko and Flajolet & Vallée independently discovered a discretization over [0, 1] of the action of the Gauss map on certain Taylor expansions centered at $x = 1/2$, the result of which is a discrete matrix M with entries of the form

$$M_{i,j} = \frac{(-1)^i}{i!\,(-2)^j} \sum_{n=0}^{j} \binom{j}{n} (-2)^n\,(n+2)_i \left[\zeta(n+i+2)\left(2^{n+1+2} - 1\right) - 2^{n+i+2}\right],$$

where $(x)_i = \Gamma(x+i)/\Gamma(x)$ is a so-called Pochhammer symbol and where $\zeta(z)$ denotes Riemann's zeta function, whose second-largest (in absolute value) eigenvalue λ_1 is precisely the value λ above. These and other methods can be found in the work of Briggs, as well as in the literature published by Finch, MacLeod, and Plouffe. It is unknown whether λ is irrational or transcendental.

GaussKuzminWirsingOperator

Let $\mathbb{V}$ be the Banach space of functions analytic in the disk $\{z : |z-1| < 3/2\}$ and continuous in its closure, equipped with the supremum norm. The Gauss-Kuzmin-Wirsing operator $\mathcal{L}$ is defined for $f \in \mathbb{V}$ through

$$\mathcal{L}[f(t)](z) = \sum_{m=1}^{\infty} \frac{1}{(z+m)^2} f\left(\frac{1}{z+m}\right).$$

$\mathcal{L}$ is a nuclear trace class operator of order 0. The eigenvalues λ_n, $n \in \mathbb{Z}^+$ of $\mathcal{L}$ are simple and real with alternating sign and with $\lambda_1 = 1$, $|\lambda_{n+1}| \leq |\lambda_n|$, and $\sum_{n=1}^{\infty} |\lambda_n|^{\varepsilon}$ for every $\varepsilon > 0$. Asymptotically

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = -\phi^2.$$

$\mathcal{L}$ has the following properties:

$$\mathrm{Tr}(\mathcal{L}) = \int_0^{\infty} \frac{J_1(2\,x)}{e^x + 1}\, dx$$

$$\mathrm{Tr}\left(\mathcal{L}^2\right) = \int_0^{\infty}\int_0^{\infty} \frac{J_1\left(2\sqrt{x\,y}\right)^2}{(e^x+1)(e^y+1)}\, dx\, dy.$$

GaussMap

The Gauss map τ is defined as

$\tau : \mathbb{R} \to \mathbb{Z}$

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

GaussMapFixedPoints

Let τ be the Gauss map

$\tau : \mathbb{R} \to \mathbb{Z}$

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

The fixed points of the Gauss map are the numbers

$$\xi_n = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{n} = \frac{\sqrt{n^2+4} - n}{2},$$

where $n \in \mathbb{Z}^+$.

GaussMapIntegral

Let τ be the Gauss map

$\tau : \mathbb{R} \to \mathbb{Z}$

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

The following integral holds:

$$\int_0^1 \tau(x)\, dx = \gamma - 1.$$

GaussMapInverse

Let τ be the Gauss map

$\tau : \mathbb{R} \to \mathbb{Z}$

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Let $0 < \xi < 1$ and let

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

be a regular continued fraction representation of ξ. Then the inverse τ^{-1} of the Gauss map is given by

$$\tau^{-1}(\xi) = \left\{ \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{m\, \delta_{k,0} + \left(1 - \delta_{k,0}\right) b_{k+1}} : m \in \mathbb{Z}^+ \right\} = \left\{ \frac{1}{\xi + m} : m \in \mathbb{Z}^+ \right\}.$$

GaussMapIsErgodic

The Gauss map is ergodic for the Gauss measure.

GaussMapRepresentation

Let τ be the Gauss map

$\tau : \mathbb{R} \to \mathbb{Z}$

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Let $0 < \xi < 1$ and let

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

be a continued fraction and A_n / B_n the sequence of its convergents. Then

$$\xi = \frac{A_n + \tau^n(\xi)\, A_{n-1}}{B_n + \tau^n(\xi)\, B_{n-1}}.$$

GaussMeasure

Given the measurable space $(\mathbb{R}, L)$ where L denotes the σ-algebra of Lebesgue-measurable subsets of $\mathbb{R}$, the Gauss measure is defined to assign to each set $A \in L$ the value $\mu(A)$, where

$$\mu(A) = \frac{1}{\ln(2)} \int_A \frac{d\lambda}{1 + x}$$

for λ the usual Lebesgue measure on $\mathbb{R}$.

GeneralContinuedFractionContraction

Let

$$\xi = b_0 + \mathop{\mathrm{K}}_{k=1}^{N} \frac{a_k}{b_k}$$

be a continued fraction and p_k/q_k the sequence of its convergents. The contin-
ued fraction (called contraction)

$$\eta = \beta_0 + \mathop{\mathrm{K}}_{k=1}^{M} \frac{\alpha_k}{\beta_k}$$

with convergents P_k/Q_k, where

$$\frac{P_k}{Q_k} = \frac{p_{n_k}}{q_{n_k}}$$

for $n_0 < n_1 < n_2 < \ldots$ has the numerators and denominators α_k, β_k where

$$\beta_0 = \frac{p_{n_0}}{q_{n_0}}$$

$$\alpha_1 = (-1)^{n_0} \left(\prod_{j=1}^{n_0+1} a_j\right) \frac{q_{n_1-n_0-1,n_0+1}}{q_{n_0}}$$

$$\beta_1 = q_{n_1}$$

$$\alpha_k = (-1)^{n_{k-1}-n_{k-2}-1} \left(\prod_{j=n_{k-2}+2}^{n_{k-1}+1} a_j\right) q_{n_{k-2}-n_{k-3}-1,n_{k-3}+1}\, q_{n_k-n_{k-1}-1,n_{k-1}+1}$$

$$\beta_k = q_{n_k-n_{k-2}-1,n_{k-2}+1}$$

and $n_{-1} = -1$. Here $p_{n,m}\big/q_{n,m}$ are the convergents of the continued fraction

$$b_m + \mathop{\mathrm{K}}_{j=1}^{n} \frac{a_{m+j}}{b_{m+j}}.$$

GeneralizedContinuedFraction

There are no fewer than two distinct continued fraction concepts described as generalized continued fraction.

Perhaps most commonly, a numerical continued fractions ξ is described as “generalized” provided ξ is of the form

$$\xi = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

where the partial numerators $a_1, a_2, \ldots$ are allowed to be arbitrary. This is in contrast to the case where $a_k = 1$ for $k = 1, 2, \ldots$, whereby the resulting continued fraction is considered regular.

At least one other source defines a generalized continued fraction to be any continued fraction with elements consisting of arbitrary mathematical objects such as vectors in $\mathbb{C}^n$, $\mathbb{C}$-valued square matrices, Hilbert space operators, multivariate expressions, other continued fractions, etc. As it is written, a numerical continued fraction can be used to construct one of these generalized fractions in the following way: Given a continued fraction of the form

$$\xi = b_0 + \mathrm{K}(a_n / b_n)$$

with associated second-order recursion $A_n = b_n A_{n-1} + a_n A_{n-2}$, $B_n = b_n B_{n-1} + a_n B_{n-2}$, $n = 1, 2, 3, \ldots$, subject to the initial conditions $B_{-1} = 0$, $A_0 = b_0$, $A_{-1} = B_0 = 1$, define an nth order recursion among the elements of ξ. The result of this will be a continued fraction $\hat{\xi}$ which is said to be generalized due to the fact that each of the approximants $\hat{A}_n / \hat{B}_n$ of $\hat{\xi}$ are n-dimensional vectors rather than numerical constants.

GeneralizedGaloisPeriodicRegularContinuedFraction

Let $\xi > 1$ be an irrational solution of a quadratic equation with rational coefficients of the form

$$\xi = \frac{P + \sqrt{D}}{Q}$$

with p, Q, $D \in \mathbb{Z}$ with $P \geq 0$, $D > 0$, and $Q > 0$, and $Q \mid \left(D - P^2\right)$. If its conjugate

$$\eta = \frac{P - \sqrt{D}}{Q}$$

then ξ has periodic regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with $b_{n+j} = b_n$ for $n \geq n_0$ with

$n_0 = 0$ if $-1 < \eta < 0$

$n_0 = 1$ if $0 < \eta < 1$

$n_0 \geq 1$ if $\eta > 1$.

GeneralizedGaussKuzminTheorem

Let

$$g = \phi^{-1}$$

and

$$G = \phi,$$

T_g be a generalized Gauss map, $U : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$U(x,\ y) = \left\{T_g(x),\ \frac{1}{\left\lfloor g^2 + \frac{1}{x} \right\rfloor + y \operatorname{sgn}(x)}\right\},$$

λ be the Lebesgue measure on $\mathbb{R}^2$, $J(x,\ y) = (0,\ x) \times (0,\ y)$, $m_n(x,\ y)$ be given by

$$m_n(x,\ y) = \lambda\left(\left(U^n\right)^{-1} (J(x,\ y))\right).$$

Then

$$m_n(x,\ y) = \frac{\ln\left(\frac{1 + x\, y}{1 - g^2\, y}\right)}{\ln(G)} + O(g^n).$$

GeneralizedKhinchinConstantLaw

Let $0 < \xi < 1$ be an irrational number with the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then for almost all ξ and $p < 1$, $p \neq 0$ the following p-dependent limit exists

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} b_k^p \right)^{1/p} = K_p$$

and is a fixed constant.

GeneralizedKhinchinConstantOfGeneralizedGaussMap

Let T_k, $k \in (-\infty, -1) \bigcup (0, \infty)$ be the generalized Gauss map

$$T_k(x) = \frac{1}{k \frac{x}{1-x}} - \left\lfloor \frac{1}{k \frac{x}{1-x}} \right\rfloor.$$

Then for some $\xi \in (0, 1)$ the generalized regular continued fraction expansion

$$\xi = \underset{j=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_j}$$

can be obtained through

$$b_j = T_k^j(\xi).$$

Then for almost all $\xi \in [0, 1]$,

$$\lim_{n\to\infty} \sqrt{\prod_{j=1}^{n} \begin{cases} b_j + 1 & \text{for } b_j > 0 \\ |b_j| & \text{for } b_j < 0 \end{cases}} = \prod_{j=1}^{\infty} \left(\frac{(j+|k|)^2}{(j+|k|)^2 - 1} \right)^{\operatorname{sgn}(k) \ln(j) / \ln\left(\left|\frac{k+1}{k}\right|\right)}.$$

GeneralizedKhinchinConstantOfGeneralizedRenyiMap

Let T_k, $k \in (-\infty, -1) \bigcup (0, \infty)$ be the generalized Gauss map

$$T_k(x) = \frac{1}{k \frac{x}{1-x}} - \left\lfloor \frac{1}{k \frac{x}{1-x}} \right\rfloor.$$

Then for some $\xi \in (0, 1)$ the generalized regular continued fraction expansion

$$\xi = \underset{j=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_j}$$

can be obtained through

$$b_j = T_k^j(\xi).$$

Then for almost all $\xi \in [0, 1]$

$$\lim_{n\to\infty} \sqrt{\prod_{j=1}^{n} \begin{cases} b_j + 1 & \text{for } b_j > 0 \\ |b_j| & \text{for } b_j < 0 \end{cases}} = \prod_{j=1}^{\infty} \left(\frac{(j + |k|)^2}{(j + |k|)^2 - 1} \right)^{\operatorname{sgn}(k) \ln(j) / \ln\left(\left|\frac{k}{k-1}\right|\right)}.$$

GeneralizedKhinchinLevyTheoremOfGeneralizedGaussMap

Let T_k, $k \in (-\infty, -1) \bigcup (0, \infty)$ be the generalized Gauss map

$$T_k(x) = \frac{1}{k \frac{x}{1-x}} - \left\lfloor \frac{1}{k \frac{x}{1-x}} \right\rfloor.$$

Then for some $\xi \in (0, 1)$ the generalized regular continued fraction expansion

$$\xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

with convergents A_n / B_n can be obtained through

$$b_j = T_k^j(\xi).$$

Then for almost all $\xi \in [0, 1]$

$$\lim_{n\to\infty} \ln\left(\frac{|B_n|}{n}\right) = \begin{cases} \ln\left(\sqrt{|k|}\right) - \frac{1}{\ln\left(\sqrt{\frac{|k|+1}{|k|}}\right)} + \mathrm{Li}_2\left(-\frac{1}{|k|}\right) & \text{for } k > 0 \\ \ln\left(\sqrt{|k|}\right) - \frac{1}{\ln\left(\sqrt{\frac{|k|+1}{|k|}}\right)} + \mathrm{Li}_2\left(\frac{1}{|k|}\right) & \text{for } k < 0. \end{cases}$$

GeneralizedKhinchinLevyTheoremOfGeneralizedRenyiMap

Let T_k, $k \in (-\infty, -1) \bigcup (0, \infty)$ be the generalized Rényi map

$$T_k(x) = \frac{1}{k\,\frac{x}{1-x}} - \left\lfloor \frac{1}{k\,\frac{x}{1-x}} \right\rfloor.$$

Then for some $\xi \in (0, 1)$ the generalized regular continued fraction expansion

$$\xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

with convergents A_n / B_n can be obtained through

$$b_j = T_k^j(\xi).$$

Then for almost all $\xi \in [0, 1]$

$$\lim_{n\to\infty} \ln\left(\frac{|B_n|}{n}\right) = \begin{cases} \ln\left(\sqrt{|k|}\right) - \frac{1}{\ln\left(\sqrt{\frac{|k|+1}{|k|}}\right)} + \mathrm{Li}_2\left(-\frac{1}{|k|}\right) & \text{for } k < 0 \\ \ln\left(\sqrt{|k|}\right) - \frac{1}{\ln\left(\sqrt{\frac{|k|+1}{|k|}}\right)} + \mathrm{Li}_2\left(\frac{1}{|k|}\right) & \text{for } k > 0. \end{cases}$$

GeneralRotationRelationForFiniteRegularContinuedFractions

Let ξ be a finite regular continued fraction

$$\xi = b_0 + \underset{\kappa=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_k}.$$

Let k, l, m, $n \in \mathbb{Z}^+$ and $k < l < m < n$. Then the following identity holds:

$$\left(\prod_{j=k}^{l-1}\left(b_j + \underset{\kappa=j+1}{\overset{n}{\mathrm{K}}} \frac{1}{b_\kappa}\right)\right) \times \left(\prod_{j=m+1}^{n}\left(b_j + \underset{\kappa=1}{\overset{j-l}{\mathrm{K}}} \frac{1}{b_{j-\kappa}}\right)\right) =$$

$$\left(\prod_{j=m+1}^{n}\left(b_j + \underset{\kappa=1}{\overset{j-k}{\mathrm{K}}} \frac{1}{b_{j-\kappa}}\right)\right)\left(\prod_{j=k}^{l-1}\left(b_j + \underset{\kappa=j+1}{\overset{m}{\mathrm{K}}} \frac{1}{b_\kappa}\right)\right).$$

GeometricInterpretationOfInefficientContinuedFractionSequences

Let ξ be an integer continued fraction,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n},$$

r_n be the continued fraction convergent of ξ, and λ_n be a consecutive subsequence of b_n. Then $|b_n| \geq 2$ and there does not exist λ_n which is inefficient is equivalent to r_n being a Farey geodesic.

GloballyAnalyticHypoellipticAreNotNecessarilyGloballyHypoelliptic

Let α be the irrational whose continued fraction $a_n = 10^{-n!}$. Then

$$V = d/\mathrm{dx} - \alpha\, d/\mathrm{dy}$$

is globally analytic hypoelliptic but not globally hypoelliptic.

GoldenRatio

The oft-studied golden ratio ϕ has a number of equivalent definitions framed in a variety of different contexts. Historically, the golden ratio is defined to be the unique number x for which a rectangle of side ratio $1 : x$ can be divided into a unit square and a separate rectangle whose side ratio is also $1 : x$, i.e., it is the division of a given length into two parts such that the ratio of the shorter to the longer equals the ratio of the longer part to the whole. Therefore, ϕ is the unique positive real number for which the identity

$$\frac{\phi}{1} = \frac{1}{\phi - 1.}$$

The constant ϕ and its various properties have been studied since antiquity with various constructions attributed to Euclid and Pythogoras, among others.

Simplifying the above identity, ϕ is thus the unique positive real number for which $\phi^2 = \phi + 1$. Dividing both sides by ϕ yields $\phi = 1 + 1/\phi$ and thereby yields a recursive definition of ϕ whose first few terms have the form

$$\phi = 1 + \frac{1}{\phi} = 1 + \frac{1}{1 + \frac{1}{\phi}} = 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{\phi}}} \cdots.$$

As this suggests, ϕ is the unique real number whose regular continued fraction has the form $\phi = [1; 1, 1, 1, \ldots]$ or, in Gauss notation,

$$\phi = 1 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{1}.$$

Solving the above equation algebraically yields the exact value of ϕ, namely $\phi = \left(1 + \sqrt{5}\right)/2$ which, to fifty decimal places, is equal to

$$\phi = 1.61803398874989484820458683436563811772030917980576 \ldots .$$

In addition to the above, one can find a vast number of connections between ϕ and the theory of continued fractions. For example, it is a well-known fact that $\left\{\phi^{n+1}\right\}_{n=-1}^{\infty}$ and $\left\{\psi^{n+1}\right\}_{n=-1}^{\infty}$ are both solutions to the three-term recurrence relation $X_n = X_{n-1} + X_{n-2}$, $n = 1, 2, 3, \ldots$, where ϕ is as above and where $\psi = \left(1 - \sqrt{5}\right)/2$ is the second solution of the equation $x^2 - x - 1 = 0$, and because $\left\{\phi^{n+1}\right\}$ and $\left\{\psi^{n+1}\right\}$ are $\mathbb{C}$-linearly independent elements, they form a basis of the vector space $\mathfrak{L}$, the solution space of the recurrence relation above and a degree 2 vector space over $\mathbb{C}$. Moreover, because the canonical partial numerators $\{A_n\}_{n=1}^{\infty}$, respectively partial denominators $\{B_n\}_{n=1}^{\infty}$, of an arbitrary continued fraction $\xi = \mathrm{K}(a_n/b_n)$ are also elements of $\mathfrak{L}$, it follows that A_n and B_n are $\mathbb{C}$-linear combinations of ϕ^{n+1} and ψ^{n+1} for any arbitrary continued fraction $\xi = \mathrm{K}(a_n/b_n)$, $n = 1, 2, 3, \ldots$. Among the significant ramifications of this are the so-called Binet $ formula, as well as a multitude of significant literature in areas ranging from operator theory to algebraic field theory and beyond.

GoodBirthRateForContinuedFractions

Let q be a real number where $0 \leq q \leq 1$,

$$\xi(q) = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a(n)(q)}{b_n(q)}$$

be a generalized continued fraction, $f(q)$ be the birth-death process from continued fraction of $\xi(q)$, k be a positive real, and $C(q)$ be a positive real. Then given $b_n < k$, $0 \leq \frac{d\sqrt{a(n)(q)}}{dq} \leq C(q)$, and $0 \leq -\frac{db_n(q)}{dq} \leq C(q)$, it follows that $\exists_{0 \leq q_1 \leq 1} (f(q) \text{ is good} \Leftrightarrow q \leq q_1)$.

GoodBirthRateForRogersRamanujanContinuedFractions

Let q be a real number where $0 \leq q \leq 1$, ξ be the Rogers Ramanujan continued fraction of q, and λ_n be positive reals of its associated birth-death process, i.e., where

$\lambda_0 = 1$

and

$\lambda_{n-1}(1 - \lambda_n) = q^n$.

Then $\exists_{0 \leq q_1 \leq 1} \forall_{\{0 < q < q_1, n\}} \lambda_n > 0$.

GraggWarnerHenriciPflugerBounds

Let

$$\xi = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a generalized continued fraction whose convergents are denoted w_n and $a_n > 0$ and $\mathrm{Re}(b_n) > 0$. Set

$$\alpha_n = \frac{a_n}{\mathrm{Re}(b_{n-1})\,\mathrm{Re}(b_n)}.$$

Then for all $m \geq n$,

$$|w_m - w_{n-1}| < 2\,\alpha_1 \prod_{i=2}^{n} \frac{\sqrt{4\,\alpha_i + 1} - 1}{\sqrt{4\,\alpha_i + 1} + 1}.$$

GraphPropertiesAssocatedWithHypocycloidConvergents

Consider the closed hypocycloid S of q cusps whose parameterized form is given by

$$S(t) = \begin{cases} x(t) = (\theta - 1)\, r \cos(t) + r \cos((\theta - 1)\, t) \\ y(t) = (\theta - 1)\, r \sin(t) + r \sin((\theta - 1)\, t) \end{cases}$$

for $0 < \theta = p/q < 1$ and let $\xi = [0; b_1, b_2, \ldots]$ denote the simple continued fraction corresponding to t with convergents $\xi_n = A_n / B_n$, $n = 0, 1, 2, \ldots$. Then the sequence $\{|B_n\, t - A_n|\}_{n=0}^{\infty}$ decreases to zero as $n \to \infty$, whereby it follows that the convergents ξ_n correspond to nearly equally-spaced sets of B_n cusps in the graph of S. Moreover, because

$$B_n\, |B_n\, t - A_n| < \frac{B_n}{B_{n+1}} \leq \frac{1}{b_{n+1}}$$

for $n = 0, 1, 2, \ldots$ and because cusps of S "clump" for near-minimum values of $B_n\, |B_n\, t - A_n|$, it follows that large values of the partial quotients b_n of ξ also result in cusp "clumping" for the graph of S.

HallTheorem

Hall's theorem says that any real number t can be decomposed into a sum of the form

$$t = n + [0; b_1, b_2, \ldots] + [0; b_1^*, b_2^*, \ldots]$$

where $n \in \mathbb{Z}$ and where $1 \leq b_k, b_k^* \leq 4$ for $k = 1, 2, 3, \ldots$. Named after mathematician Marshall Hall, Hall's theorem is meant to provide the set $\mathbb{R}$ of real numbers an analogue of a certain decomposition property Cantor's middle thirds set C, namely that C satisfies the identity

$$C + C = I + I$$

where $I = [0, 1]$. Though difficult, Hall's original paper provides details on a slew of continued fraction constructions and properties; Rockett and Szüsz provide a second, more concrete elaboration.

HamburgerAssociatedSeriesConvergenceTheorem

Let

$$S = \sum_{k=1}^{\infty} \frac{c_k}{z^k}$$

be a formal power series with coefficients c_k such that for all n there exist constants M and ρ so that

$$|c_n| \leq M \frac{(n-1)!}{\rho^{n-1}}$$

holds and the Hankel determinant of the $c_1, c_2, \ldots$

$$C_n = \begin{vmatrix} c_1 & c_2 & \ldots & c_{n-1} & c_n \\ c_2 & c_3 & \ldots & c_n & c_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} & c_n & \ldots & c_{2n-3} & c_{2n-2} \\ c_n & c_{n+1} & \ldots & c_{2n-2} & c_{2n-1} \end{vmatrix}$$

is positive for all n. Then the associated Perron continued fraction of S with variable z converges uniformly in any part of the complex z-plane that does not contain the real axis.

HankelDeterminant

Given a formal power series of the form $f_0(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$, the corresponding Hankel determinants H_k, $k = 0, 1, \ldots$, have the form $H_0 = 1$ and

$$H_k = \begin{vmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_1 & c_2 & \cdots & c_k \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-1} & c_{k-2} & \cdots & c_{2k-2} \end{vmatrix}.$$

HarmanWongConvergentsNumeratorDenominatorProperty

Let the set of positive integers $\{c_1, c_2 .... \}$ be called an acceptable set if

$\gcd(c_j, c_{j+1}, m) = 1$ for $1 < j < n - 1$

$c_{j+2} = c_j \bmod \gcd(c_{j+1}, m)$ for $1 < j < n - 2$

for a positive integer m. (If the set is of length 2, $\{c_1, c_2\}$ it is acceptable if $\gcd(c_1, c_2, m) = 1$; all sets of length 1 are acceptable.) Let ξ be an irrational nonalgebraic real number with regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents numerators A_n and B_n.

For almost all ξ, there are infinitely many j such that

$A_{j+i} = c_i \pmod{m}$ for $1 < i < n$

and

$B_{j+i} = c_i \pmod{m}$ for $1 < i < n$.

If the set of positive integers $\{c_1, c_2 .... \}$ is not acceptable, then there are no solutions for

$A_{j+i} = c_i \pmod{m}$ for $1 < i < n$

and

$B_{j+i} = c_i \pmod{m}$ for $1 < i < n$

for any ξ.

HarmanWongDenominatorValueMeasure

Let S_m be the set of positive integers $\{c_1, c_2, \ldots, c_m\}$.

Let ξ be an irrational nonalgebraic real number with regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

The number of matches $C_{S_m,N}$

$$a_{j+i} = c_i$$

for $j \leq N$ is for almost all ξ asymptotically

$$C_{S,N} \sim (\mu(\rho) - \mu(\sigma))\, N$$

where

$$\mu(x) = \log_2(x+1)$$

and

$$\rho = \begin{cases} 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{c_k + \delta_{k,n}} & \text{for } m \text{ even} \\ 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{c_k} & \text{for } m \text{ odd} \end{cases}$$

$$\sigma = \begin{cases} 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{c_k} & \text{for } m \text{ even} \\ 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{c_k + \delta_{k,n}} & \text{for } m \text{ odd.} \end{cases}$$

HaydenConvergenceTheorem

Let ξ be the continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{\begin{cases} 1 & \text{for } k = 1 \\ a_k & \text{for } k > 1 \end{cases}}{1}$$

with the sequence of convergents A_k / B_k. If there exist constants $s > 0$ and $q > 0$ and $0 < r < 1$, such that

$$|a_{3n-1}| \geq (1 + q + s)^2$$

$$|a_{3n}| \geq r\, q$$

$$|a_{3n+1}| \geq r\, s$$

then the sequence of convergents

$$\alpha_k = \frac{A_{\left(1-(-1)^k + 6k\right)/4}}{B_{\left(1-(-1)^k + 6k\right)/4}}$$

converges.

HaydenRegionSequenceConvergenceTheorem1

Let $V = \{V_1, V_2, \ldots\}$ of regions of the complex plane where each V_n is the form $\{z : |z| \leq R_n\}$ or $\{z : |z| \geq R_n\}$ for $R_n \in \mathbb{R}$.

If for every $p > 1$, V_p or V_{p+1} is bounded and there exist sequences of numbers $0 < g_n < 1$ and $0 < r_n \leq 1$ such that

$$\begin{cases} |z| \leq r_n\, g_n (1 - g_{n-1}) & \text{if } V_n \text{ is bounded} \\ |z| \geq (2 - g_n) & \text{if } V_n \text{ is unbounded} \end{cases}$$

and if $P = \{p_1, p_2, \ldots\}$ are all indices of the sequence V such that V_{p_k} is unbounded and P is either finite or $\prod_{j=1}^{\infty} r_{p_j} = 0$, then for any sequence of complex numbers $a_k \in V_k$, the continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{\begin{cases} 1 & \text{for } k = 1 \\ a_k & \text{for } k > 1 \end{cases}}{1}$$

converges.

HaydenRegionSequenceConvergenceTheorem2

Let $V = \{V_1, V_2, \ldots\}$ of regions of the complex plane where each V_n is the form $\{z : |z| \leq R_n\}$ or $\{z : |z| \geq R_n\}$ for $R_n \in \mathbb{R}$.

If for every $p > 1$, V_p or $V_{p+1}$is bounded and there exists a sequence of numbers $0 < g_n < 1$ such that

$$\begin{cases} |z| \leq g_n (1 - g_{n-1}) & \text{if } V_n \text{ is bounded} \\ |z| \geq (2 - g_n) & \text{if } V_n \text{ is unbounded} \end{cases}$$

and if

$$\sum_{r=1}^{\infty} \left(\prod_{j=1}^{r} \left(\begin{cases} |z| \leq \frac{g_n}{1-g_n} & \text{if } V_n \text{ is bounded} \\ |z| \geq \frac{2-g_n}{1-g_n} & \text{if } V_n \text{ is unbounded} \end{cases} \right) \right) < \infty$$

then for any sequence of complex numbers $a_k \in V_k$, the continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{\begin{cases} 1 & \text{for } k = 1 \\ a_k & \text{for } k > 1 \end{cases}}{1}$$

converges absolutely.

HaydenRegionSequenceDivergenceTheorem1

Let $V = \{V_1, V_2, \ldots\}$ of regions of the complex plane where each V_n is the form $\{z : |z| \leq R_n\}$ or $\{z : |z| \geq R_n\}$ for $R_n \in \mathbb{R}$.

If there exists an integer $p > 1$ such that both, V_p and V_{p+1} are unbounded, then there exists a sequence of complex numbers $a_k \in V_k$ such that the continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{\begin{cases} 1 & \text{for } k = 1 \\ a_k & \text{for } k > 1 \end{cases}}{1}$$

diverges.

HaydenRegionSequenceDivergenceTheorem2

Let $V = \{V_1, V_2, \ldots\}$ of regions of the complex plane where each V_{3n-1} is the form $\{z : |z| \geq s\}$ where $s > 0$, each V_{3n-1} is the form $\{z : |z| \leq 1\}$ where $s > 0$, and each V_{3n+1} is the form $\{z : |z| \leq 1\}$.

Then there exists a sequence of complex numbers $a_k \in V_k$ such that the continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{\begin{cases} 1 & \text{for } k = 1 \\ a_k & \text{for } k > 1 \end{cases}}{1}$$

diverges.

HigherOrderKhinchinConstants

One constant that comes up regularly in the study of the ergodic theory of regular continued fractions is Khinchin's constant K. However, $K = K_0$ is merely one of an infinite family of Hölder means K_p, $p < 1$, associated to regular continued fractions. Indeed, let $\xi = [b_0; b_1, b_2, \ldots]$ be a regular continued fraction and define for each $p < 1$, $p \neq 0$, the limit

$$K_p = \lim_{n\to\infty}\left\{\frac{1}{n}\left(b_1^p + b_2^p + \ldots + b_n^p\right)^{1/p}\right\}.$$

This value, which is an almost everywhere constant independent of ξ or n, is called the pth order Khinchin constant or the Khinchin constant of order p. The "standard Khinchin constant" is then defined to be the limiting case $K_0 = \lim_{p\to 0} K_p$.

The collection K_p possesses many unique and well-studied properties. For example, when $p < 1$ is nonzero, it can be shown that K_p has the almost everywhere equivalent forms

$$K_p = \left(\frac{1}{\ln(2)}\sum_{i=1}^{\infty} i^p \ln\left(1 + \frac{1}{i(i+2)}\right)\right)^{1/p} = \left(\frac{1}{\ln(2)}\int_0^1 \frac{(\lfloor 1/t\rfloor)^p}{1+t}\, dt\right)^{1/p}$$

and that K_0 has analogous expressions of the form

$$K_0 = \prod_{i=1}^{\infty}\left(1 + \frac{1}{i(i+2)}\right)^{\ln(i)/\ln(2)} = \exp\left(\frac{1}{\ln(2)}\int_0^1 \frac{\lfloor 1/t\rfloor}{1+t}\, dt\right)$$

almost everywhere. In addition to their obvious ties to the theory of continued fractions, the family of Khinchin means plays a significant role in the theories of polylogarithm and computing.

HillamThronConvergenceCorollary

Let ξ be the continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k}.$$

Then ξ converges if and only if there exists a $c \in \mathbb{C}$ and $r \in \mathbb{R}$ with $|c| < r$ such that for all $n \geq 1$,

$0 < |a_n| \leq (r - |c|)(|b_n + c| - r)$.

HillamThronConvergenceTheorem

Let ξ be the continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k}.$$

Let K be the disk $\{z : |z - c| \leq r\}$ with $|c| < r$. If

$$t_n(z) \subset K$$

where

$$t_n(z) = \frac{a_n}{b_n + z}$$

for all $n \geq 1$ and $a_n \neq 0$, then ξ converges and $\xi \in K$.

HurwitzContinuedFractionCoprimeConvergentIdentity

Let x be an irrational number, ξ be the Hurwitz continued fraction expansion of x, A_n be the convergent numerator of ξ, and B_n be the convergent denominator of ξ. Then

$$A_n B_{n-1} - A_{n-1} B_n = (-1)^{n+1}.$$

ImproperlyEquivalent

Two complex numbers ξ, $\eta \in \mathbb{C}$ are called improperly equivalent if there exists an improperly unimodular map m such that $\eta = m(\xi)$.

ImproperlyUnimodularMap

A unimodular map m is called improperly unimodular if $\det(m) \in \{\pm i\}$.

IndependentAndIdenticallyDistributedBernoulliRandomContinuedFractionsMarkovChainConvergesToNonatomicProbability

Let Z_n be an independent identically distributed Bernoulli random variable, P its probability expectation, and X_n a Markov chain defined by

$$X_n = 1 / X_{n-1} + Z_n.$$

Then X_n converges to a singular probability π invariant under the Gauss map which is nonatomic.

IndependentAndIdenticallyDistributedBernoulliRandomContinuedFractionsMarkovChainConvergesToNonatomicProbabilityWithFullSupport

Let Z_n be an independent identically distributed Bernoulli random variable, P its probability expectation, and X_n a Markov chain defined by

$X_n = 1/X_{n-1} + Z_n$.

Then $X_n$converges to a singular probability π invariant under the Gauss map which is nonatomic.

InequalitiesForHausdorffDimensionForBoundedPartialQuotients

Let E be a subset of the natural numbers less than or equal to n, $E(R)$ be the regular continued fractions ξ whose partial denominators lie in E, and H be the Hausdorff dimension. Then given $n \geq 8$,
$1 - 4/(n \ln(2)) < H(E(R)) < 1 - 1/(8\, n \ln(n))$.

InfiniteContinuedFractionsAreIrrational

Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be a regular continued fraction. Then given $b_n > 0$ for all $n > 0$, it follows that ξ is irrational.

InfiniteQuadraticSurdsWithGivenContinuedFractionPeriod

For any $k > 0$, there are an infinite number of squarefree positive integers N whose continued fraction of $\sqrt{N}$ has period k.

InfiniteSumContinuedFraction

Let ξ be a positive irrational number with continued fraction expansion

$$\frac{1}{\xi} = b_0 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

with $a_j \in \mathbb{Z}^+$ and convergents A_n / B_n (with $B_{-1} = 0$).

For integer $m \geq 1$, define

$$S_m(\xi) = (m-1) \sum_{j=1}^{\infty} m^{-\lfloor j\,\xi \rfloor}.$$

Then

$$S_m(\xi) = t_0 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{t_j},$$

where

$$t_0 = m\, a_0$$

$$t_n = \frac{m^{B_n} - m^{B_{n-2}}}{m^{B_{n-1}} - 1}.$$

InfiniteSumContinuedFractionConvergents

Let ξ be a positive irrational number with continued fraction expansion

$$\frac{1}{\xi} = b_0 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

with $a_j \in \mathbb{Z}^+$ and convergents p_n/q_n (with $q_{-1} = 0$). For integer $m \geq 1$, define

$$S_m(\xi) = (m-1) \sum_{j=1}^{\infty} m^{-\lfloor j\,\xi \rfloor}$$

with convergents P_n/Q_n and

$$T_m(\xi) = (m-1) \sum_{j=1}^{\infty} m^{-\mathrm{fl}(j\,\xi)}$$

where $\mathrm{fl}(x) = \lfloor x \rfloor$ for noninteger x and $\mathrm{fl}(x) = x - 1$ for integer x.

Then

$$P_n = \begin{cases} \sum_{j=1}^{p_n} m^{q_n - \mathrm{fl}(j\,q_n/p_n)} & \text{for } n \text{ even} \\ \sum_{j=1}^{p_n} m^{q_n - \lfloor j\,q_n/p_n \rfloor} & \text{for } n \text{ odd} \end{cases}$$

$$Q_n = \frac{m^{q_n} - 1}{m - 1}$$

and

$$\frac{P_n}{Q_n} = \begin{cases} T_m\left(\frac{q_n}{p_n}\right) & \text{for } n \text{ even} \\ S_m\left(\frac{q_n}{p_n}\right) & \text{for } n \text{ odd.} \end{cases}$$

InvariantMeasureOfGeneralizedGaussMap

Let T_k, $k \in (-\infty, -1) \bigcup (0, \infty)$ be the generalized Gauss map

$$T_k(x) = \frac{1}{k\,\frac{1-x}{x}} - \left\lfloor \frac{1}{k\,\frac{1-x}{x}} \right\rfloor.$$

Then the invariant measure μ_k of T_k on the interval $[0, 1]$ is given by

$$\mu_k(x) = \frac{\mathrm{sgn}(k)}{\ln\left(\left|\frac{k+1}{k}\right|\right)} \frac{1}{x+k}.$$

T_k is ergodic with respect to μ_k.

InvariantMeasureOfGeneralizedRenyiMap

Let T_k, $k \in (-\infty, 0) \bigcup (1, \infty)$ be the generalized Rényi map

$$T_k(x) = \frac{1}{k^{\frac{x}{1-x}}} - \left\lfloor \frac{1}{k^{\frac{x}{1-x}}} \right\rfloor.$$

Then the invariant measure μ_k of T_k on the interval $[0, 1]$ is given by

$$\mu_k(x) = \frac{\operatorname{sgn}(k)}{\ln\left(\left|\frac{k}{k-1}\right|\right)} \frac{1}{x+k-1}.$$

T_k is ergodic with respect to μ_k.

InversionSymmetry

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction. Then the following identity holds:

$$\frac{1}{\xi} = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{\begin{cases} 1 & \text{for } k = 1 \\ a_{k-1} & \text{for } k \geq 2 \end{cases}}{\begin{cases} b_0 & \text{for } k = 1 \\ b_{k-1} & \text{for } k \geq 2. \end{cases}}$$

IrrationalPeriodicityTheoremForDExpansions

Let x be a real number, D be a measurable subset of $[0, 1]$, and ξ

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n}$$

be the D-expansion continued fraction for x. Then x is irrational if and only if a_n and b_n are periodic.

IteratedGaussMapDifference

Let τ be the Gauss map

$\tau : \mathbb{R} \to \mathbb{Z}$

$$\tau(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

Let $0 < \xi < 1$ be an irrational number with regular continued fraction representation

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

and A_n / B_n the sequence of its convergents. Then

$$\xi - \underset{k=1}{\overset{n}{\mathrm{K}}} \frac{a_k}{b_k} = \frac{(-1)^n \tau^n(\xi)}{B_n(B_n + \tau^n(\xi) B_{n-1})}.$$

IteratedLinearFractionalTransformation

Much of the literature agrees that the first connection between linear fractional transformations and the theory of continued fractions is due to the work of Weyl.

On a technical level, there are a variety of ways to define a continued fraction which formalize the intuitive case of fractional representations of real numbers and one of the most fundamental ways of doing so is by way of an iteration of a specific linear fractional transformation. Given an ordered pair $(\{a_m\}_{m\in\mathbb{Z}^+}, \{b_m\}_{m\in\mathbb{Z}^*})$ of complex sequences for which $a_m \neq 0$ for $m \geq 1$, define the sequences $\{s_n(w)\}_{n\in\mathbb{Z}^*}$, $\{S_n(w)\}_{n\in\mathbb{Z}^*}$ so that $s_0(w) = b_0 + w$,
$s_n(w) = a_n (b_n + w)^{-1}$ for $n = 1, 2, 3, \ldots$, $S_0(w) = s_0(w)$, and
$S_n(w) = S_{n-1}(s_n(w))$,
$n = 1, 2, 3, \ldots$. By way of a simple substitution, it follows that, for $n = 1, 2, 3, \ldots$, the approximant function $S_n(w)$ has the form

$$S_n(w) = (s_0 \circ s_1 \circ s_2 \circ \cdots \circ s_n)(w),$$

or equivalently,

$$S_n(w) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}.$$

Thus, evaluating S_n at $w = 0$ yields the finite generalized continued fraction ξ of the form

$$\xi = S_n(0) = b_0 + \mathop{\mathrm{K}}_{m=1}^{n} \frac{a_m}{b_m}.$$

One of the benefits of using this particular nomenclature when defining continued fractions is that defining related concepts like convergence, e.g., is a matter of a very simple notational extension: In particular, one could use the above definition to say that the sequence ξ_n of convergents converges to an infinite continued fraction ξ precisely when $\xi = \lim_{n\to\infty} S_n(0)$. This definition is used throughout the book by Cuyt et al. and is relatively prevalent among continued fraction literature. More details can also be found in the 1970 article by Mandell and Magnus.

IteratedLogarithmLaw

For a collection $\{Y_n\}$ of identically distributed independent random variables with means $\mu_n = 0$ and variances $\text{var}(Y_n) = 1$, the iterated logarithm law says that with probability 1,

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2\, n \ln(\ln(n))}} = 1 \text{ and } \liminf_{n\to\infty} \frac{S_n}{\sqrt{2\, n \ln(\ln(n))}} = -1,$$

where $S_n = Y_1 + \cdots + Y_n$. The application of this concept to continued fractions is a result of the correspondence between the theory of power series related to random walks and the continued fraction representations of these power series.

IteratedLogarithmLawForNumberOfPartialQuotients

Let $k_n(x)$ denote the exact number of partial quotients in the regular continued fraction expansion $x = [b_o; b_1, b_2, \ldots]$ which can be obtained by considering the first n decimals of x. Then for almost all $x \in (0, 1)$, there exists a constant $\sigma > 0$ for which

$$\limsup_{n\to\infty} \frac{k_n(x) - \frac{6 \ln(2) \ln(10)}{\pi^2} n}{\sigma \sqrt{2\, n \ln(\ln(n))}} = 1 \text{ and } \liminf_{n\to\infty} \frac{k_n(x) - \frac{6 \ln(2) \ln(10)}{\pi^2} n}{\sigma \sqrt{2\, n \ln(\ln(n))}} = -1.$$

JacobiPerronAlgorithmTheoremInNDimensions

Let x be a real vector in n dimensions. Then the Jacobi Perron algorithm of x produces a sequence of integral vectors $a_k(n)$ where

$$\lim_{k\to\infty} [\text{angle between } x \text{ and } a_k(n)] = 0.$$

JacobiPerronAlgorithmTheoremInTwoDimensions

Let x be a real two-dimensional vector. Then the Jacobi Perron algorithm of x produces a sequence of integral vectors $a_k(n)$ where

$$\lim_{k\to\infty} [\text{angle between } x \text{ and } a_k(n)] = 0.$$

JacobiSymbolsOfConvergentsOfRegularContinuedFractionExpansion

Let $\xi \in \mathbb{R}\,\mathbb{Q}$ have the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

with convergents A_k / B_k. Then $\left(\frac{A_k}{B_k}\right)$ depends only on the residue classes $\overline{b_0}, \overline{b_1}, \ldots, \overline{b_k}$, where $\overline{b_k} \equiv b_k \bmod 4$.

JacobiSymbolsOfRegularContinuedFractionExpansionOfE

Consider the regular continued fraction expansion of e

$$e = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

with convergents A_k / B_k. Then

$$\left(\frac{A_{k+24}}{B_{k+24}}\right) = \left(\frac{A_k}{B_k}\right)$$

for all k. (Jacobi symbols that are not defined are treated as being equal.)

JonesThronConditionsForContinuedFractionCorrespondenceToLaurentSeries

Let

$$\xi(z) = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n(z)}{b_n(z)}$$

be a generalized continued fraction, P_n be the formal Laurent series satisfying

$$P_n = a_{n+1}\, P_{n+2} + b_n\, P_{n+1},$$

$$L = \frac{P_0}{P_1}$$

be a formal Laurent series set, and λ denote the Laurent exponent. Then given $\lambda(b_{n-1}) + \lambda(b_n) < \lambda(a_n)$ and $\lambda(b_{n-1}) + \lambda\left(\frac{P_n}{P_{n+1}}\right) < \lambda(a_n)$, it follows that $\xi(z)$ corresponds to the Laurent series L.

KhinchinConstant

Named for its discoverer, Khinchin's is a constant is a real number K defined to be the almost-everywhere asymptotic bound of the geometric means of the partial quotients of an arbitrary real number. Said differently, given a real number x with corresponding regular continued fraction $\xi = [b_0; b_1, b_2, \ldots]$, let $G_n(x)$ denote the geometric mean of the first n partial quotients of ξ, i.e.,

$G_n(x) = (b_1 \cdot b_2 \cdots b_n)^{1/n}$.

Khinchin proved that for almost all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} G_n(x) = K$$

where K is a constant independent of n or x. It is unknown currently whether Khinchin's constant K is irrational or transcendental, though to 50 decimal places, K can be computed to equal

$K = 2.6854520010653064453097148354817956938203822939944 6\ldots$.

Moreover, while it is known that nearly every real number has a regular continued fraction, the geometric mean of whose partial quotients approach K asymptotically, no such $x \in \mathbb{R}$ has been exhibited; on the other hand, several significant real numbers have been shown to have regular continued fractions which do not approach K, among which are $x = e$, $x = \sqrt{2}$, $x = \sqrt{3}$, and $x = \phi$, where ϕ denotes the golden ratio. The regular continued fraction of K starts out $K = [2; 1, 2, 5, 1, 1, 2, 1, 1, \ldots]$.

Khinchin's derivation of the above-mentioned result is actually a corollary deduced from the proof of a much stronger result. In particular, he showed that if $f(r)$ is a non-negative function defined on all $r \in \mathbb{Z}^+$ and if there exist positive constants C and δ for which $f(r) < C r^{-\delta} \sqrt{r}$, $r = 1, 2, \ldots$, then for almost all real numbers $x \in (0, 1)$ with associated regular continued fraction $\xi = [0; b_1, b_2, \ldots]$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(b_k) = \sum_{r=1}^{\infty} f(r) \frac{\ln\left\{1 + \frac{1}{r(r+2)}\right\}}{\ln 2.}$$

From this more general statement, Khinchin's constant can be derived by defining $f(r) = \ln r$, whereby the above equation can be rewritten as

$$\lim_{n\to\infty} \sqrt[n]{b_1 \cdot b_2 \cdots b_n} = \prod_{r=1}^{\infty} \left\{1 + \frac{1}{r(r+2)}\right\}^{\ln r/\ln 2},$$

where the infinite product converges to K almost everywhere. As Khinchin himself notes, the phrasing of the original result is general enough to allow for an entire slew of interest results concerning probability densities related to continued fraction element distribution, etc., though he also notes that no analogue to the geometric mean result can be formulated for the arithmetic mean.

KhinchinConstantLaw

Let $0 < \xi < 1$ be an irrational number with the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then the following identity holds for almost all ξ

$$\lim_{n\to\infty} \left(\prod_{k=1}^{n} b_k \right)^{1/n} = K,$$

where K is a fixed constant.

KhinchinDiamondVaalerTheorem

Let F_t be a positive arithmetical function, ϵ be a positive real, α be an irrational number where $0 \le \alpha \le 1$, ξ be a half-regular continued fraction of α,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n},$$

$$S_N(F, \alpha) = \sum_{n=1}^{N} F_{b(n)},$$

and $c(N, \alpha, F)$ be a positive real where $0 \le c(N, \alpha, F) \le 1$. Then given

$$\exists_\epsilon \frac{\sum_{j=1}^{N} \frac{F_j^2}{j^2}}{\left(\sum_{j=1}^{N} \frac{F_j}{j^2} \right)^2} \le N \ln^{-3/2-\epsilon}(N),$$

it follows that for almost all α

$$S_N(F, \alpha) = \max_{1 \le n \le N} F_{b(n)}\, c(N, \alpha, F) + \frac{1 + o(N)}{\ln(2)} \sum_{i=1}^{\infty} F_i \ln\left(1 + \frac{1}{i\,(2+i)}\right).$$

LagrangeQuadraticIrrationalyTheorem

Let ξ be a quadratic irrational, meaning a nonrational solution of a quadratic equation with rational coefficients. Then the regular continued fraction representation of ξ,

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

is ultimately always periodic.

LambdaSubQFractionsHaveTheApproximationProperty

A number field is said to have the approximation property if for every "irrational" α,

$$\left| \alpha - \frac{P}{Q} \right| < \frac{1}{k\, Q^2}$$

is satisfied by infinitely many rational elements P/Q of the number field and k is a positive fixed constant.

The algebraic number field generated by

$$\lambda_q = 2 \cos\left(\frac{\pi}{q}\right)$$

for q an odd positive number ≥ 3 has the approximation property.

LaneWallCharacterization

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction and A_n/B_n the sequence of its convergents. Let

$$\sum_{n=1}^{m} \left| \frac{A_{n+1}}{B_{n+1}} - \frac{A_{n-1}}{B_{n-1}} \right| < \infty.$$

Then the continued fraction ξ converges if and only if its Stern-Stolz series diverges.

LaplaceTransformOfDurationOfExcursionByOccupationProcess

Let Λ_t be an excursion of occupation process,

$\theta(p) = \inf(t > 0,\ \Lambda_t = C)$

be a duration of excursion for Λ_t with $C > 0$ and $p > 0$, and let θ^* be the Laplace transform of θ. Then the continued fraction θ^* is an S-fraction and

$$\theta^*(p) = \frac{(C+1)\,\Phi(p,\ C+p+2,\ u)}{(C+p+1)\,\Phi(p,\ C+p+1,\ u)},$$

where Φ is the Kummer function.

LebesgueMeasureOfRegularContinuedFractionsWithGivenInitialPartialDenominators

Let $0 < \xi < 1$ have the regular continued fraction expansion

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

The Lebesgue measure λ of all ξ in $[0, 1]$ that have the initial partial denominators $b_1,\ b_2,\ \ldots\ ,\ b_n$ and where the partial denominator b_{n+1} has the value j is

$$\lambda = \begin{cases} \frac{1}{j(j+1)} & \text{for } j = 1 \\ \frac{s_n+1}{(s_n+j)\,(s_n+j+1)} & \text{for } j > 1, \end{cases}$$

where

$$s_n = \underset{k=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_{n-k+1}}.$$

LeightonConjecture

Let the C-fraction

$$\xi(z) = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{a_j\, z^{\alpha_j}}{1}$$

where $a_j \in \mathbb{C} \setminus 0$ and $\alpha_n \in \mathbb{Z}^+$ and

$$\lim_{n\to\infty} \alpha_n = \infty$$

$$\lim_{n\to\infty} |a_n|^{1/n} = 1.$$

Then $\xi(z)$ converges in the disk $\mathbb{D} = \{z : |z| < 1\}$ to a function $f(z)$ meromorphic in $\mathbb{D}$ and the boundary of $\mathbb{D}$, is the natural boundary of meromorphicity for $f(z)$.

LevyConstant

The so-called Lévy constant is intimately connected with the Khinchin constant K which provides an almost everywhere asymptotic bound on the geometric means of successive partial quotients for an arbitrary real number $x \in \mathbb{R}$ with regular continued fraction expansion ξ. In particular, given x and ξ as above with $\xi_n = A_n / B_n$ the nth convergent of ξ, the almost everywhere bound of $\sqrt[n]{B_n}$ by a constant (indeed, Khinchin proved that there exist two absolute constants a, A with $1 < a < A$ for which $a < \sqrt[n]{B_n} < A$ for almost all x and sufficiently large n) leads naturally to the question of convergence: Might one be able to compute an expected value for $\sqrt[n]{B_n}$, and might one also be able to determine an associated law of large numbers for this quantity?

At around the same time as Khinchin's works, Lévy published affirmative results to both the above questions: In particular, he showed that for

$$\gamma = \frac{\pi^2}{12 \ln(2)}$$

and for all sufficiently large n,

$$\left| \frac{1}{n} \ln(B_n) - \gamma \right| \leq \epsilon(n)$$

for almost all $t \in [0, 1$, where $\epsilon(n)$ is any positive function decreasing to zero as $n \to \infty$ for which $\sum_{n=1}^{\infty} 1 \big/ \left(\epsilon^2(n) \cdot n^2\right)$ converges. Said differently, Lévy proved that $\sqrt[n]{B_n} \to \exp(\gamma)$ as $n \to \infty$. The constant e^γ, which to 50 decimal places is equal to

$$e^\gamma = 3.2758229187218111597876818824538438636084755259823 7 \ldots ,$$

is now known as Lévy's constant. Worth noting, however, is that the phrase "Lévy's constant" sometimes refers to other related quantities depending on the author: In particular, some authors use it to denote the exponent $\gamma = \pi^2 \big/ (12 \ln(2))$, which still other authors call the Khinchin-Lévy constant. As a result, some caution must be exercised.

Both the properties possessed by and the proof which derives the Lévy constant yield as corollaries many significant results which are of interest in their own right. For example, Khinchin proved as a corollary of his version of the derivation that almost all numbers $\alpha \in \mathbb{R}$ satisfies a more general analogue of the continued fraction approximation property, while still others were able to derive the same result using a variety of measure-theoretic techniques involving ergodic theory and the solution space $\mathfrak{L}$ of a specific family of three-term recurrence relations. In a seemingly unrelated application, Corless was able to show that for an arbitrary real number x, the so-called Lyapunov exponent λ of the Gauss map G evaluated at x has the form

$$\lambda(x) = 2\,\gamma = \int_0^1 \frac{\ln(1/x)}{\ln(2)\,(1+x)}\, d\mu$$

where μ denotes regular Lebesgue measure and where γ is the exponent of the Lévy constant; he also derived an analogous formula for the Khinchin constant K, namely

$$\ln(K) = \int_0^1 \frac{\ln(\lfloor 1/x \rfloor)}{\ln(2)\,(1+x)}\, d\mu.$$

Many other results related to the Lévy constant can be found in the works of Khinchin, Lévy, Finch, Corless, Rockett, and Szüsz, among others.

LimitPeriodicContinuedFractionInequality1

Let $\xi = \mathrm{K}(b_n/1) = [0; b_1, b_2, \ldots]$ be a limit periodic continued fraction, let $b \neq 0$ be the complex number $b = \lim_{n\to\infty} b_n$ chosen so that $|\arg(b + 1/4)| < \pi$ and $\mathrm{Re}\left(\sqrt{1/4 + b}\right) > 0$, and suppose that for $n \geq 1$,

$$|b_n - b| \leq \min\left\{\frac{1}{2}\left(\left|\frac{1}{4} + b\right| + \frac{1}{4} - |b|\right), \frac{|b|}{2}\right\}.$$

Then

$$\left|\frac{\xi - S_n\left(\sqrt{b + \frac{1}{4}} - \frac{1}{2}\right)}{\xi - S_n(0)}\right| \leq 2\, d_n\, \frac{|b| + \left|b + \sqrt{b + \frac{1}{4}} + \frac{1}{2}\right|}{|b|\left(-|b| + \left|b + \frac{1}{4}\right| + \frac{1}{4}\right)},$$

where $S_n(0) = A_n/B_n$ is the nth approximant of ξ, $S_n(w) = \frac{A_n + A_{n-1}\, w}{B_n + B_{n-1}\, w}$ is the approximant function for all complex numbers w, and $d_n = \max_{m \geq n} |a_m - a|$.

LimitPeriodicContinuedFractionInequality2

Consider a sequence b_n, $n = 1, 2, \ldots$, of strictly positive real numbers and let $f(z) = \mathrm{K}(b_n\, z/1) = [0; b_1\, z, b_2\, z, \ldots]$ be a convergent limit periodic S-fraction which tends to $b = \lim b_n > 0$ as $n \to \infty$. Then, for all complex values z with $|\arg(z)| < \pi/2$,

$$\left| \frac{f(z) - S_n\left(\sqrt{b\, z + \frac{1}{4}} - \frac{1}{2}\right)}{f(z) - S_n(0)} \right| \le \frac{4\, d_n}{|x_1|\, D} \cdot \frac{4\, |z| \max\limits_{m \ge n} |b_m - b|}{b\, |z| + \mathrm{Re}\left(\sqrt{b\, z + \frac{1}{4}}\right) - \left|b\, z + \frac{1}{4}\right| - \frac{1}{4}},$$

where $S_n(0) = A_n(z)/B_n(z) =$ is the nth approximant of $f(z)$, $S_n(z) = \frac{A_n + A_{n-1}\, z}{B_n + B_{n-1}\, z}$ is the approximant function for all complex z, $d_n = \max_{m \ge n} |b_m - b|$, and x_1 is the solution of $x^2 + x - a = 0$ for which $D = |x_1 + 1| - |x_1| > 0$.

LimitsOfPeriodicCDuallyRegularFractionsAreQuadraticIrrationals

Every periodic C-dually regular continued fraction ξ converges to an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ which is quadratic over $\mathbb{Q}$.

LimitsOfPeriodicCRegularFractionsAreQuadraticIrrationals

Every periodic C-regular continued fraction ξ converges to an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ which is quadratic over $\mathbb{Q}$.

LimitsOfRamanujanQSeries

Define $K_n(q)$ as the generalized continued fraction for $|q| > 1$

$$K_n(q) = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{q^k}{1}$$

and let $R(x)$ be the Rogers-Ramanujan continued fraction and $K(x)$ be

$$K(q) = \frac{q^{1/5}}{R(q)}.$$

Then

$$\lim_{j\to\infty} K_{2\,j+1}(q) = \frac{1}{K\left(-\frac{1}{q}\right)}$$

and

$$\lim_{j\to\infty} K_{2\,j}(q) = \frac{K\left(\frac{1}{q^4}\right)}{q}.$$

LiouvilleAlgebraicIndependence

Let α_i be a real,

$$\xi_i = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{a(N,\, i)}$$

be the regular continued fraction of α_i, with convergents $p(N,\, i)/q(N,\, i)$, r be a real, f_i be a real-valued sequence with

$$\lim_{i\to\infty} f_i = \infty,$$

and N_i define a subsequence of natural numbers.

Then given $r > 1$ such that

$$\forall \left(i \geq 1 \bigwedge n \geq j \geq 1,\ a(N_{i+1},\ j) \geq q(N_i,\ 1)^{f_i}\right)$$

$$\forall \left(i \geq 1 \bigwedge n \geq j \geq 2,\ q(N_i,\ j-1) \geq r^{f_i}\, q(N_i,\ j)\right)$$

$$\forall \left(i \geq 1 \bigwedge n \geq j \geq 2,\ q(N_i+1,\ j-1) \geq r^{f_i}\, q(N_i+1,\ j)\right),$$

the α_i are algebraically independent.

LiouvilleAlgebraicIndependenceCorollary2

Let α_i be a real number,

$$\xi_i = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{a(N,\ i)}$$

be the regular continued fraction of α_i, g_i be real numbers,

$$\lim_{i\to\infty} g_i = \infty,$$

and N_i be the subsequence of positive integers. Then given $\tau > 1$, $r > 1$,

$\forall\,(N \geq 1,\ a(N+1,\ 1) \geq a(N,\ 1)^{\tau})$

$\forall\,(N \geq 1 \bigwedge n \geq j \geq 1,\ a(N,\ j-1) \geq r\, a(N,\ j))$

$\forall\,(i \geq 1 \bigwedge n \geq j \geq 2,\ a(N_i+1,\ j-1) \geq a(N_i,\ 1)^{g_i}),$

the α_i are algebraically independent.

LiouvilleAlgebraicIndependenceCorollary3

Let g_i be integers, ξ be the regular continued fraction of β, g be a non-negative integer, n be a positive integer, β be an irrational number, g_i be a real number, and define

$$S_{g_i}(\beta) = (g_i - 1) \sum_{\nu=1}^{\infty} {g_i}^{-\lfloor \beta\, \nu \rfloor}.$$

Then given $g_i \geq 2$ with distinct values and ξ has bounded partial quotients, $S_{g_i}(\beta)$ are algebraically independent.

LiouvilleContinuedFractionTheorem

Let α be an algebraic real number and ξ be its regular continued fraction with partial denominator b_n, and B_n its convergent denominator, and let d be the algebraic degree of α. Then there exists a $C > 0$ such that for all integer $n \geq 1$, $b(n) < C\, B(n)^{d-2}$.

LochsConstant

There are no fewer than two distinct constants attributed to Lochs. The first and by far most popular is derived as part of Lochs 'theorem concerning the asymptotic relation between the decimal and regular continued fraction expansions of arbitrary real numbers x. Proved in the 1960s, Lochs 'theorem says that for (Lebesgue) almost all real numbers x for which $m(x, n)$ regular continued fraction "digits" (i.e., partial quotients) needed to determine n decimal digits,

$$\lim_{n\to\infty} \frac{m(x, n)}{n} = \frac{6 \ln(2) \ln(10)}{\pi^2}.$$

The above limit, sometimes denoted $\mathfrak{L}_{L_0}$, is what is most widely-acknowledge to be Lochs 'constant; to 50 decimal places,

$\mathfrak{L}_{L_0} = 0.97027011439203392574025601921001083378128470478516 \cdots$.

Numerically, $\mathfrak{L}_{L_0}$ indicates that 100 decimal digits of every real number $x \in \mathbb{R}$ can be unambiguously determined for every 97.02 ... partial quotients of the regular continued fraction $\xi(x)$ associated to x with the exception of a set of (Lebesgue) measure zero.

This definition is remarkable in that the asymptotic limit $\mathfrak{L}_{L_0}$ is absolutely constant and hence is independent of the real number $x \in \mathbb{R}$ in question. Because of its significance, modifying and generalizing Lochs 'proof has been at the heart of a great deal of literature. For example, Lochs 'theorem was proved by Bosma, Dajani and Kraaikamp to be a specific case of the so-called Shannon-McMillan-Breiman theorem characterizing the asymptotic behavior of the measure-theoretic properties of an ergodic transformation S with respect to its entropy $h(S)$. Additional results relating $\mathfrak{L}_{L_0}$ with the theory of entropy and transformations, see the works of Kraaikamp, Billingsley, and Nakada. Moreover, $\mathfrak{L}_{L_0}$ has been shown to be intimately connected to the works of both Khinchin and Lévy and to the eponymous constants K and e^{γ}, respectively.

As mentioned initially, there is no apparent agreement on which constant should be attributed to Lochs. Indeed, some literature refers to the multiplicative reciprocal $\mathfrak{L}_{L_0}^{-1}$ of the above-mentioned constant (which is also equal to two times the base-10 logarithm of Lévy's constant e^{γ}) as Lochs 'though, of the two, $\mathfrak{L}_{L_0}$ appears to be the more common choice.

LochsTheorem

Let x be an irrational number where $0 < x < 1$ and

$d_n(x) = 10^{-n} \lfloor 10^n x \rfloor$

$e_n(x) = 10^{-n} (\lfloor 10^n x \rfloor + 1)$

be decimal approximations of x, m be a Lebesgue measure set,

$$x = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{a_n}$$

be the regular continued fraction of x,

$$d_n(x) = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_1(n)}$$

be the regular continued fraction of $d_n(x)$,

$$e_n(x) = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_2(n)}$$

be the regular continued fraction of $e_n(x)$, and

$k_n(x) = \sup(\{i : \forall\, i \le n,\ b_1(i) = b_2(i)\})$.

Then

for almost all x, $\lim_{n \to \infty} \frac{k_n}{n} = \frac{6 \ln(2) \ln(10)}{\pi^2}$.

LorentzenConditionsForContinuedFractionCorrespondenceToLaurentSeries

Let

$$\xi(z) = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n(z)}{b_n(z)}$$

be a generalized continued fraction, X_n be the formal Laurent series where

$X_n = a_n X_{n-2} + b_n X_{n-1}$,

$$L = -\frac{X_0}{X_{-1}}$$

be a formal Laurent series, and λ denote the Laurent exponent. Then given $\lambda(b_{n-1}) + \lambda(b_n) < \lambda(a_n)$ and $\lambda(b_n) < \lambda\left(\frac{X_n}{X_{n-1}}\right)$, it follows that $\xi(z)$ corresponds to L.

LowerBoundForBestRationalApproximation

Let α be a rational number where $0 \leq \alpha \leq 1$, ξ be the regular continued fraction of α, A_n be the convergent numerator of ξ, and B_n be the convergent denomina· tor of ξ. Then $|A_n - \alpha B_n| \geq \frac{1}{2 B_{1+n}}$.

LowerBoundForLyapunovExponentsOfGaussMap

Let $G(x)$ denote the Gauss map defined piecewise as

$$G(x) = \begin{cases} x & \text{for } x = 0 \\ x - \lfloor x \rfloor & \text{for } x \neq 0, \end{cases}$$

and for an arbitrary real number γ, let

$$\lambda(\gamma) = \lim_{n \to \infty} \frac{1}{n} \ln\left(\prod_{i=0}^{n} |G'(\gamma_i)|\right)$$

denote Lyapunov exponent of the orbits of the Gauss map (provided the limit exists) where $\gamma_0 = G(\gamma)$, $\gamma_{k+1} = G(\gamma_k)$ for $k = 1, 2, \ldots$, and G' denotes the derivative of G in the usual sense. Under this construction, no orbit of the Gauss map has Lyapunov exponent smaller than $\lambda(1/\phi) = 2 \ln \phi$.

LowerBoundPeriodsForNonSchinzelQuadratics

For an integer X, let

$$d(X) = A^2 X^2 + 2 B X + C$$

be a polynomial, A, B, C be integers,

$$\Delta = B^2 - A^2 C$$

$$\delta = \frac{\Delta}{\gcd(A B)^2}$$

$$x = \sqrt{d(X)}$$

be quadratic irrational numbers, ξ be the regular continued fraction of x, and $l(X)$ be the regular continued fraction period of ξ. Given $A > 0$ and $\left(4 \gcd\left(A^2 B\right)^2\right) \bmod \Delta \neq 0$, then $l(X) \geq 1 + 2 \ln\left(\sqrt{d(X)}\right) / \ln(\delta)$.

LubinskyCounterexampleToGeneralPadeConjecture

Let $H_q(z)$ be a Rogers Ramanujan continued fraction where

$q = e^{4 i \pi / \left(99+\sqrt{5}\right)}$.

Then $H_q(z)$ is a counterexample to the Padé conjecture.

LyapunovExponent

Let $G(x)$ denote the Gauss map which is defined piecewise as

$$G(x) = \begin{cases} x & \text{for } x = 0 \\ x - \lfloor x \rfloor & \text{for } x \neq 0. \end{cases}$$

For an arbitrary real number γ, the Lyapunov exponents λ of the orbits of the Gauss map are defined as

$$\lambda(\gamma) = \lim_{n \to \infty} \frac{1}{n} \ln\left(\prod_{i=0}^{n} \left|G'(\gamma_i)\right|\right)$$

provided the limit exists, where $\gamma_0 = G(\gamma)$, $\gamma_{k+1} = G(\gamma_k)$ for $k = 1, 2, \ldots$, and where G' denotes the derivative of G in the usual sense. Conceptually, the Lyapunov exponent can be thought of as the average rate of separation between the orbits of points which are initially close as they are iterated under the Gauss map.

MarkovTheorem

Given a Borel measure σ on $\mathbb{R} - [A, B]$ with Chebyshev continued fraction ξ, then ξ converges uniformly on compact sets to the Markov function associated to σ.

MarkovTheoremForRationalPerturbationsOfMarkovFunctions

Let $A < B$ and

$D = \mathbb{C} - [A, B]$

be a domain, r be a complex rational function, σ be a positive Borel measure set, $\hat{\sigma}(z)$ be the Markov function of σ,

$f = r(z) + \hat{\sigma}(z)$

be a meromorphic function, $f_n(z)$ be the Padé approximants diagonals, and g be a chordal metric on the Riemann sphere. Then given $D[\sigma] > 0$ almost everywhere in $[A, B]$, it follows that $f_n(z)$ converges uniformly on D in the chordal metric on the Riemann sphere.

Mediant

The mediant of two rational numbers $a/b < c/d$ is defined to be the rational number $(a+c)/(b+d)$. By observation, the mediant can be seen to satisfy

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

MeromorphicExtensionsOfCertainJFractions

Let $f(z)$ be a J-fraction of the form

$$f(z) = \frac{1}{z + b_0 - \frac{a_1}{z + b_1 - \frac{a_2}{z + b_2 - \frac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose without loss of generality that $\lim a_n = 1/4$, $\lim b_n = 0$. Assume, too, that for some $R > 1$,

$$\sum_{j=1}^{\infty} \left(\left| \frac{a_j - 1}{4} \right| + \left| b_j \right| \right) R^j < \infty,$$

and let $\omega = \omega(z)$ denote the transformation

$$\omega(z) = \frac{1}{2} \left((z+1)^{1/2} - (z-1)^{1/2} \right)^2$$

for all $z \in \mathbb{C}^* = \mathbb{C} \backslash [-1, 1]$ under the assumption that the roots of ω are strictly positive for $z > 1$. Under these hypotheses, $f(z)$ can be extended to a meromorphic function on all of $\mathbb{C}^{**}$ where $\mathbb{C}^{**}$ is the complete 2-sheeted Riemannian surface obtained by analytic extension of ω from $\mathbb{C}^*$ across $[-1, 1]$ into a second copy of $\mathbb{C}^*$.

MonotoneBehaviorOfEvenAndOddContinuedFractionConvergents

Let $\xi = [0; b_1, b_2, \ldots]$ be a continued fraction (either finite or infinite) which converges to some number α and let A_n / B_n denote its nth convergent, $n = 1, 2, \ldots$. Then the sequence $\{A_{2n-1} / B_{2n-1}\}_{n=1}^{\infty}$ of odd convergents of ξ increase to α and the sequence $\{A_{2n} / B_{2n}\}_{n=1}^{\infty}$ of even convergents decrease to α.

MuellerContinuedFraction

Given real numbers p and q, let

$$C = \frac{x^p (1-x)^{q-1} \Gamma(p+q)}{\Gamma(p+1)\Gamma(q)}$$

$$\mu(s) = \frac{q-s}{p+s}$$

$$b_n = \begin{cases} 1 & \text{for } n = 1 \\ -\frac{x(p+s-1)(p+s)\mu(s)}{(1-x)((p+2s-2)(p+2s-1))} & \text{for } n = 2s \\ \frac{s x (p+q+s)}{(1-x)((p+2s-1)(p+2s))} & \text{for } n = 2s+1. \end{cases}$$

Then the continued fraction

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

converges to

$$\xi = \frac{\mathrm{B}_x(p, q)}{C\,\mathrm{B}(p, q)}.$$

MultidimensionalContinuedFraction

A multidimensional continued fraction is an extension of the notion of continued fraction representations of real numbers to n-tuples $(a_1, a_2, \ldots, a_n)$ in $\mathbb{R}^n$, $n > 1$. First proposed in 1839 by Hermite, the idea of generalizing real continued fractions to higher dimensions has been the focus of a considerable amount of literature. It should come as no surprise, then, that the phrase "multidimensional continued fraction" exists in a variety of contexts as penned by many different authors; a few of those expositions are summarized here.

One of the earliest attempts at such a generalization is due to Jacobi who, in

1868, published an algorithm for computing so-called ternary continued fractions $[(p_1, q_1); (p_2, q_2); \ldots]$ whose elements (p_k, q_k) are all ordered pairs of real numbers. More precisely, Jacobi's algorithm associates to triples $u_1, v_1, w_1 \in \mathbb{R}$ of real numbers a continued fraction of the form

$$\left(\frac{v_1}{u_1}, \frac{w_1}{u_1}\right) = [(p_1, q_1); (p_2, q_2); (p_3, q_3); \ldots]$$

whose nth convergents $(B_n / A_n, C_n / A_n)$ satisfy the four-term recurrence relations

$$A_n = q_n A_{n-1} + p_n A_{n-2} + A_{n-3},$$

$$B_n = q_n B_{n-1} + p_n B_{n-2} + B_{n-3},$$

$$C_n = q_n C_{n-1} + p_n C_{n-2} + C_{n-3},$$

where $u_{n+1} = v_n - p_n u_n$, $v_{n+1} = w_n - q_n u_n$, $w_{n+1} = u_n$, $p_n = \lfloor v_n / u_n \rfloor$, and $q_n = \lfloor w_n / u_n \rfloor$. The upshot of Jacobi's method is that it possesses many obvious properties analogous to the case of standard continued fraction representations of real numbers. On the other hand, Jacobi's algorithm left much to be desired, most notably the fact that many observable patterns were largely unprovable at the time.

Since then, many different, largely more general notions of multidimensional continued fractions have been devised. One of the more well-known of these is due to Szerkeres, who devised an algorithm whereby sequences $[b_1, b_2, \ldots]$ of positive integers called continued k-fractions are associated with k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of real numbers via a rather in-depth set theoretic construction. Like Jacobi's, Szerkeres 'algorithm yields a highly-analogous continued fraction theory. For example, Cusick's exposition on the Szerkeres algorithm illustrates the process of defining sets of integer k-tuples, respectively $(k+1)$-tuples

$$A(n, j) = \left(A^{(1)}(n, j), \ldots, A^{(k)}(n, j)\right),$$

respectively

$$(B(n, 0), B(n, 1), \ldots, B(n, k)),$$

manipulations of which produce nth approximations $P_n / Q_n = A(s_n, 0) / B(s_n, 0)$ for the k-fraction $[b_1, b_2, \ldots]$ of $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ which satisfy the identity

$$\lim_{n\to\infty} \frac{A^{(i)}(s_n, 0)}{B(s_n, 0)} = \alpha_i$$

for each $i = 1, 2, \ldots, k$ where, here, $s_n = \sum_{k=1}^n b_k$, $n = 1, 2, 3, \ldots$. This identity is the multidimensional analogue of the fact that $\lim_{n\to\infty} A_n / B_n = \alpha$ for real one-dimensional continued fractions ξ with nth convergents $\xi_n = A_n / B_n$. More details of this particular construction can be found in Cusick and its references.

Still another popular exposition is due to Schweiger, who approaches the construction via matrices rather than sequences. In particular, Schweiger defines a fibered system (B, T) to be a set B and a mapping $T : B \to B$ with the property that one can partition B into sets $\{B(i) : i \in I\}$ with the property that $T|_{B(i)}$ is injective for all $i \in I$. Here, I is an indexing set which is as most countably infinite. Under this construction, (B, T) is said to be a multidimensional continued fraction (also called piecewise fractional linear) provided that $B \subset \mathbb{R}^n$

continued fraction (also called piecewise fractional linear) provided that $B \subset \mathbb{R}$ for some n and that for every "digit" $k \in I$, there exists an invertible matrix

$\alpha = \alpha(k) = \left(\left(A_{i\,j}\right)\right) \in GL(n+1, \mathbb{Z})$,

$0 \le i,\ j \le n$, such that

$$y_i = (T\,x)_i = \frac{A_{i\,0} + \sum_{i=1}^{n} A_{i\,j}\,x_j}{A_{00} + \sum_{j=1}^{n} A_{0\,j}\,x_j}$$

for every $x \in B(k) \subset \mathbb{R}^n$.

Other definitions of various depths and contexts can be found throughout the literature. A purely geometrical definition can be found in Karpenkov whose motivation lies in the related work of Klein dating back to the late 19th century. A more technically sophisticated approach centered on linear algebra and functional analysis can be found in Khanin et al. Functional multidimensional continued fractions, including branched continued fractions, are discussed in the thesis of Aryal, who also examines convergence of multidimensional continued fractions and the relationships between such fractions and so-called multiple power series. Though apparently rare, a small portion of the literature compares various multidimensional fraction constructions, e.g., Schweiger, who examines his construction and its properties relative to the constructions of Jacobi and others. For other similar resources, see the introduction of Karpenkov as well as its references.

NachreinerGuentherDeterminantFormulas

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction and A_k / B_k the sequence of its convergents.

Then the following explicit form for the numerators and denominators of the convergents holds:

$$A_n = \det \begin{pmatrix} b_0 & -1 & 0 & \ldots & 0 & 0 & 0 \\ a_1 & b_1 & -1 & \ldots & 0 & 0 & 0 \\ 0 & a_2 & b_2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{n-2} & -1 & 0 \\ 0 & 0 & 0 & \ldots & a_{n-1} & b_{n-1} & -1 \\ 0 & 0 & 0 & \ldots & 0 & a_n & b_n \end{pmatrix}$$

$$B_n = \det \begin{pmatrix} b_1 & -1 & 0 & \ldots & 0 & 0 & 0 \\ a_2 & b_2 & -1 & \ldots & 0 & 0 & 0 \\ 0 & a_3 & b_3 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{n-2} & -1 & 0 \\ 0 & 0 & 0 & \ldots & a_{n-1} & b_{n-1} & -1 \\ 0 & 0 & 0 & \ldots & 0 & a_n & b_n \end{pmatrix}.$$

NearestIntegerDistanceExceptionalLimit

There exist irrational numbers ξ with regular continued fraction expansion

$$\xi = b_0 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

and A_n / B_n the sequence of its convergents such that $\beta = m\,\alpha + n$ for all $m, n \in \mathbb{Z}^+$

$$\lim_{n \to \infty} \min(\lfloor \beta\, B_n \rfloor, \lceil \beta\, B_n \rceil) = 0.$$

NearestIntegerDistanceLimit

Let ξ be an irrational number with regular continued fraction expansion

$$\xi = b_0 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

with bounded $b_j \in \mathbb{Z}^+$ and A_n / B_n the sequence of its convergents. Let β be an irrational number. Then

$$\lim_{n\to\infty} \min(\lfloor \beta B_n \rfloor, \lceil \beta B_n \rceil) = 0$$

if and only if $\beta = m\,\alpha + n$ with $m,\ n \in \mathbb{Z}$.

NearestIntegerFractionConvergenceRate

Let α be a real, ξ be the regular continued fraction of α with convergents p_n / q_n, and ψ be the nearest integer continued fraction of α with convergents A_n / B_n. Let k_n be integers where

$$\frac{A_n}{B_n} = \frac{p_{k_n}}{q_{k_n}}.$$

Then for almost all α,

$$\lim_{n\to\infty} \frac{n}{k_n} = \frac{\ln(\phi)}{\ln(2)}.$$

NondecreasingExponentCaseOfLeightonConjecture

Let ξ be a C-fraction,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n\, z^{\alpha_n}}{1},$$

D be the unit disk, and B be the domain boundary set of D. Then given $a_n \neq 0$,

$\alpha_n \in \mathbb{Z}^+$,

$$\lim_{n\to\infty} \alpha_n = \infty$$

$$\lim_{n\to\infty} |a_n|^{1/\alpha_n} = 1$$

$$\forall_{n\geq 1} \sum_{i=1}^{n} (-1)^{-i+n}\, \alpha_i \geq 0,$$

it follows that ξ converges in D to a meromorphic function and that B is the natural meromorphic boundary.

NumeratorDenominatorDerivativeRelation

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction and p_k / q_k the sequence of its convergents. Then the following relation holds:

$$\frac{\partial p_N}{\partial b_0} = q_N.$$

NumeratorDenominatorSymmetry

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction and A / B_k the sequence of its convergents.

Let

$$\zeta = b_N + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_{N-k+1}}{b_{N-k}}$$

be a derived continued fraction and P_k / Q_k the sequence of its convergents.

Then the following identity holds for the sequences of numerators of the two convergents:

$$A_N = P_N.$$

NuttallTheorem

Let e_j be complex numbers,

$$H(z) = \frac{1}{\prod_{i=1}^{2p} (z - e_i)}$$

be a meromorphic function, R be the hyperelliptic Riemann surface set of $H(z)$ of genus $g = p - 1$, π be the hyperelliptic Riemann surface projection set of R, π_1 be the hyperelliptic Riemann surface first sheet set of R, π_2 be the hyperelliptic Riemann surface second sheet set of R, $w(z)$ be a meromorphic function where

$$w(z)^2 = H(z).$$

Let $\mathrm{dG}(z)$ be the Abelian differential of the third kind set of R,

$$u(z) = \mathrm{Re}\left(\int_{e_1}^{z} \mathrm{dG}(z)\right)$$

be the harmonic function set with domain R,

$$\Gamma = \{z \mid u(z) = 0\},$$

S be the projection of Γ composed of arcs S_j from e_{2j-1} to e_{2j}, $S^+(j)$ be the hyperelliptic Riemann surface arc above set of S_j, $w^+(z)$ be a meromorphic function

$$\forall_{x \in S^+(j)} w^+(\pi(x)) = w(x),$$

$$f(z) = \frac{1}{2\pi i} \int_{\zeta \in S} \frac{\rho(\zeta)}{(\zeta - z)\, w^+(z)} \, d\zeta$$

be a meromorphic function,

$$D = \pi(\{z : u(z) > 0\})$$

be the domain of $f(z)$, $\rho(x)$ be a holomorphic function where $\forall_{x \in S} \rho(x) \neq 0$, $\Psi_n(z)$ be a meromorphic function whose domain is $R - \Gamma$, and whose divisor is $\sum_{i=1}^{g} z_i + \pi_2(\infty)(n - g) - \pi_1(\infty)\, n$ and

$$\forall_{\zeta \in \Gamma} \rho(\zeta)\, \pi_1(\Psi_n(\zeta)) = \pi_2(\Psi_n(\zeta)),$$ and

$f_n(z)$ be the Padé approximants diagonal set for f at 0. Then

$$f(z) - f_n(z) = \frac{(1 + o(1)) \prod_{j=1}^{g} (z - z_j)}{\sqrt{H(z)}\ \Psi_n(z)^2}.$$

OddContraction

Let $\xi = b_0 + \mathrm{K}(a_m / b_m)$ be a generalized continued fraction with nth approxi-mant $\xi_n = A_n / B_n$. A continued fraction $\zeta = d_0 + \mathrm{K}(c_m / d_m)$ with nth approximant $\zeta_n = C_n / D_n$ is said to be an even contraction of ξ if and only if $\zeta_n = \xi_{2n+1}$ for $n = 0, 1, 2, \ldots$. Note that ξ has an even contraction if and only if $b_{2n+1} \neq 0$ for all positive integers n.

OstrowskiNumberSystemIntegers

Let ξ be the positive irrational number $0 < \xi < 1$ with regular continued fraction expansion

$$\xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

and convergents A_n / B_n.

For every irrational number ξ with $0 < \xi < 1$, any integer n can be uniquely written as

$$N = \sum_{k=1}^{m} c_k B_{k-1},$$

where

$0 \leq c_1 \leq b_1 - 1$

$0 \leq c_1 \leq b_1$ for $k \geq 2$

$c_k = 0$ if $c_{k+1} = b_{k+1}$.

OstrowskiNumberSystemReals

Let ξ be the positive irrational number $0 < \xi < 1$ with regular continued fraction expansion

$$\xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

and convergents A_n / B_n.

Let

$\theta_n = \xi B_n - A_n$.

For every irrational number ξ with $0 < \xi < 1$, any real x with $0 < x < 1$ can be uniquely written as

$$x = \sum_{k=1}^{m} c_k \, |\theta_{k-1}|,$$

where

$0 \le c_1 \le b_1$ for $k \ge 1$

$c_k = 0$ if $c_{k+1} = b_{k+1}$

and $c_k \ne b_k$ for infinitely many c_k.

PadeApproximant

Given a function f with associated Taylor series $A(x) = \sum_{j=0}^{\infty} a_j x^j$, the Padé approximants to f are a collection of rational approximations devised to provide accurate estimations of f by way of matching A as long as is mathematically feasible and deviating onward in order to avoid perpetuation of error. In particular, the $[L, M]$ Padé approximant to f is defined to be the rational function $P_L(x)/Q_M(x)$, where $P_L(x) = p_0 + p_1 x + \cdots + p_L x^L$ and $Q_M(x) = q_0 + q_1 x + \cdots + q_M x^M$ are polynomials of degree at most L and M, respectively, which satisfies the asymptotic relation $A(x) - P_L(x)/Q_M(x) = O\left(x^{L+M+1}\right)$.

This asymptotic relation uniquely determines the coefficients p_i and q_j, $i = 0, 1, \ldots, L$, $j = 0, 1, \ldots, M$, the association of which can be written out algorithmically as follows: Define $a_n \equiv 0$ if $n < 0$, $q_j \equiv 0$ if $j > M$, and

$$\begin{array}{lcl}
a_0 & = & p_0 \\
a_1 + a_0 q_1 & = & p_1 \\
a_2 + a_1 q_1 + a_0 q_2 & = & p_2 \\
\vdots & & \vdots \\
a_L + a_{L-1} q_1 + \cdots + a_0 q_L & = & p_L \\
a_{L+1} + a_L q_1 + \cdots + a_{L-M+1} q_M & = & 0 \\
\vdots & & \vdots \\
a_{L+M} + a_{L+M-1} q_1 + \cdots + a_L q_M & = & 0.
\end{array}$$

Note that the above procedure is what remains when the normalization assumption $Q_M(0) = 1$ is made; this is assumed in several modern contexts though is often omitted in classical literature on the subject.

PadeApproximantColumn

Given a function $f(z)$, the kth column of Padé approximants of f are the Padé approximants of f of the form

$$\frac{p_0(z)}{q_k(z)},\ \frac{p_1(z)}{q_k(z)},\ \frac{p_2(z)}{q_k(z)},\ \ldots$$

where, for integers L, $M \geq 0$,

$$\frac{p_L(z)}{q_M(z)} = \frac{p_0 + p_1\, z + \cdots + p_L\, z^L}{q_0 + q_1\, z + \cdots + q_M\, z^M}$$

denotes the $[L, M]$ Padé approximant of f. The use of the term "column" is suggestive of the fact that the collection $\{[j, k]\}$, $j = 0, 1, 2, \ldots$, forms the kth column of the Padé Table corresponding to f. Worth noting, too, is that the first column of Padé approximants of f consists precisely of the partial sums of its Taylor series expansion.

PadeApproximantDenominator

Given a function $f(z)$, the denominators of the Padé approximants of f are the polynomials $q_0(z)$, $q_1(z)$, $q_2(z)$, … where, for integers L, $M \geq 0$,

$$\frac{p_L(z)}{q_M(z)} = \frac{p_0 + p_1\, z + \cdots + p_L\, z^L}{q_0 + q_1\, z + \cdots + q_M\, z^M}$$

denotes the $[L, M]$ Padé approximant of f.

PadeApproximantDiagonal

Given a function $f(z)$, the Padé diagonal approximants are the Padé approximants of f of the form

$$\frac{p_0(z)}{q_0(z)},\ \frac{p_1(z)}{q_1(z)},\ \frac{p_2(z)}{q_2(z)},\ \ldots$$

where for N a positive integer,

$$\frac{p_N(z)}{q_N(z)} = \frac{p_0 + p_1\, z + \cdots + p_N\, z^N}{q_0 + q_1\, z + \cdots + q_N\, z^N}$$

denotes the $[N, N]$ Padé approximant of f. The use of the term "diagonal" is suggestive of the fact that the collection of all $[N, N]$ Padé approximants of f, $N = 0, 1, 2, \ldots$, forms the diagonal of the Padé table corresponding to f.

PadeApproximantNumerator

Given a function $f(z)$, the numerators of the Padé approximants of f are the polynomials $p_0(z)$, $p_1(z)$, $p_2(z)$, ... where, for integers L, $M \geq 0$,

$$\frac{p_L(z)}{q_M(z)} = \frac{p_0 + p_1 z + \cdots + p_L z^L}{q_0 + q_1 z + \cdots + q_M z^M}$$

denotes the $[L, M]$ Padé approximant of f.

PadeApproximantRow

Given a function $f(z)$, the jth row of Padé approximants of f are the Padé approximants of f of the form

$$\frac{p_j(z)}{q_0(z)}, \frac{p_j(z)}{q_1(z)}, \frac{p_j(z)}{q_2(z)}, \ldots$$

where, for integers L, $M \geq 0$,

$$\frac{p_L(z)}{q_M(z)} = \frac{p_0 + p_1 z + \cdots + p_L z^L}{q_0 + q_1 z + \cdots + q_M z^M}$$

denotes the $[L, M]$ Padé approximant of f. The use of the term "row" is suggestive of the fact that the collection $\{[j, k]\}$, $k = 0, 1, 2, \ldots$, forms the jth row of the Padé Table corresponding to f.

PadeConjecture

Let $f(z)$ be a complex-valued function defined on some domain $G \subset \mathbb{C}$ for which

$\{z \in \mathbb{C} : |z| \leq R \text{ for some } R > 1\} \subset G$

and suppose that, with the exception of M poles $z_1, z_2, \ldots, z_M$ within the disc $|z| \leq 1$ and except for at the point $z = 1$ where f is assumed continuous only when points $|z| \leq 1$ are considered, f is holomorphic on $|z| \leq 1$ with corresponding power series $F(z)$. Under these hypotheses, a subsequence of the collection

$$\frac{p_0(z)}{q_0(z)}, \frac{p_1(z)}{q_1(z)}, \frac{p_2(z)}{q_2(z)}, \ldots$$

of $[N, N]$ Padé approximants to f converges uniformly to f on the set Ω as $N \to \infty$. Here, Ω denotes the set formed by removing from the region $|z| \leq 1$ arbitrarily small open neighborhoods centered at each pole z_m.

PadeTable

Given a function $f(z)$ with $[L, M]$ Padé approximant

$$\frac{p_L(z)}{q_M(z)} = \frac{p_0 + p_1 z + \cdots + p_L z^L}{q_0 + q_1 z + \cdots + q_M z^M},$$

$L, M = 0, 1, 2, \ldots$, the so-called Padé table is a rectangular matrix consisting of L rows and M columns whose (L, M) entry is identically equal to $[L, M]$. In some literature, the Padé table used is the transpose of the table described here, i.e., it is the $M \times L$ matrix whose (M, L) entry is the $[M, L]$ approximant of the function f.

PalindromicRegularContinuedFraction

Let $p > q > 1$ and let

$$\frac{p}{q} = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}$$

be the corresponding regular continued fraction with $b_k \in \mathbb{Z}^+$. Then a necessary and sufficient condition for the existence of a palindromic expansion $b_{N-j} = b_j$ for $j = 0, 1, \ldots , N$ is

$$p \mid q^2 - 1$$

or

$$p \mid q^2 + 1.$$

PalindromicRegularContinuedFractionInfiniteRadicals

Let a_n be a palindromic string set and m be its string length. For any d be a square free integer, let

$$x_1 = \sqrt{d}$$

be a quadratic irrational,

$$\xi_1 = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n^{(1)}}$$

the regular continued fraction of x_1, and l_1 the regular continued fraction period of ξ_1. Also let

$$x_2 = \frac{1}{2}\left(\sqrt{d} + 1\right)$$

be a quadratic irrational,

$$\xi_2 = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n^{(2)}}$$

the regular continued fraction of x_2, b_n and l_2 the regular continued fraction period of ξ_2. Let X be integers d such that either $l_1 = m$ and $b_1 = a$ or $l_2 = m$ and $b_2 = a$. Then X is infinite.

ParabolaTheorem

There are a number of results of varying generalities which are known as "the parabola theorem," and while most are equivalent (or analogous, in the case of theorems in more general settings), perhaps the most geometrically-intuitive version is the one given by Voll and Lorentzen and outlined below.

Suppose $\alpha \in \mathbb{R}$ is fixed and satisfies $|\alpha| < \pi/2$ and define $E_\alpha \subseteq \hat{\mathbb{C}}$ to be the subsets

$$E_\alpha = \left\{ a \in \mathbb{C} \,:\, |a| - \mathrm{Re}\left(a\, e^{-2 i a}\right) \leq \frac{1}{2} \cos^2(\alpha) \right\}.$$

A generalized continued fraction $\xi = \mathrm{K}(a_n/1)$ for which $a_n \in E_\alpha$, $n = 0, 1, 2, \ldots$, converges to a finite value $x \in \mathbb{C}$ provided that $S(\xi) = \infty$ where here, $S(\xi) = S$ denotes the so-called Stern-Stolz series

$$S = \sum_{n=1}^{\infty} \left| \prod_{k=1}^{n} a_k^{(-1)^{n+k-1}} \right|$$

associated with ξ. Moreover, if $S < \infty$, then $\{f_{2n}\}$, $\{f_{2n+1}\}$ converge absolutely to distinct finite values and $\left\{S_{2n}^{\xi}\right\}$, $\left\{S_{2n+1}^{\xi}\right\}$ converge generally to these values.

Here, $f_n = S_n^{\xi}(0)$ and S_n^{ξ} is the Möbius transformation associated to ξ defined for all $w \in \mathbb{C}$ by the approximant function

$$S_n^{\xi}(w) = \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{\ddots + \cfrac{a_n}{1 + w}}}}.$$

While being somewhat simpler notationally, this particular statement seems at first glance to have lost the "parabola" aspect of the theorem; in reality, however, the region E_α above has a geometric boundary ∂E_α which is precisely a parabola in the complex plane.

Worth noting is that, because of its rich history, there are a variety of naming conventions regarding this theorem resulting from contributions made by a variety of authors. Indeed, it is not uncommon to see any or all of the names Gragg, Warner, Scott, Paydon, or Wall attached as prefixes. For classical sources stating and proving results related hereto, see works by Paydon, Scott, and Wall from the 1940s. In addition, many sources such as Gragg & Warner, Lorentzen, and Hovstad address various aspects of this theorem from more modern viewpoints while still others, e.g., Short, Voll, and Lorentzen & Waadeland, provide geometric interpretations of the theorem and prove theorems derived therefrom.

ParabolaTheoremEstimation

Let

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{1}$$

be a continued fraction with $a_k \neq 0$ and A_n / B_n the sequence of its convergents.
Let

$$|a_k| - \mathrm{Re}\left(a_k\, e^{-2 i \alpha}\right) \leq \frac{\cos^2(\alpha)}{2}$$

where $-\pi/2 < \alpha < \pi/2$. Then for all $n \geq 1$

$$2\,\mathrm{Re}\left(B_n\, \overline{B}_n\, e^{i \alpha}\right) - |B_{n-1}|^2 \geq \frac{2}{\cos(\alpha)}\, |B_{n-1}| \left(|B_{n-1}| - \left|B_{n-1}\, e^{-i \alpha} - B_{n-2} \cos(\alpha)\right|\right).$$

ParabolicConvergenceTheorem1

Let P_α be a certain parabola in the complex plane with focus (0, 0) going through $z = -1/4$ characterized by the fact that $b_n \in P_\alpha$ if and only if $|b_n| - \mathrm{Re}\left(b_n\, e^{-2 i \alpha}\right) \leq \cos^2(\alpha) \big/ 2$, $\alpha \in (-\pi/2, \pi/2)$. Let ξ be a continued fraction of the form $\xi = [0; b_1, b_2, \ldots]$. If $b_n \in P_0$ (that is, $b_n \in P_\alpha$ and $\alpha = 0$) for all $n = 1, 2, \ldots$ and if at least one of the series

$$\sum_{\nu=1}^{\infty} \frac{b_2\, b_4 \cdots b_{2\nu}}{b_3\, b_5 \cdots b_{2\nu+1}},\quad \sum_{\nu=2}^{\infty} \frac{b_3\, b_5 \cdots b_{2\nu-1}}{b_4\, b_6 \cdots b_{2\nu}}$$

diverges, then ξ converges to some complex number b.

ParabolicConvergenceTheorem2

Let P_α be a certain parabola in the complex plane with focus (0, 0) going through $z = -1/4$ characterized by the fact that $b_n \in P_\alpha$ if and only if $|b_n| - \mathrm{Re}\left(b_n\, e^{-2 i \alpha}\right) \leq \cos^2(\alpha) \big/ 2$, $\alpha \in (-\pi/2, \pi/2)$. Let ξ be a continued fraction of the form $\xi = [0; b_1, b_2, \ldots]$. If for all $n = 1, 2, \ldots$, $b_n \in K$ where K is a closed region contained in the interior of P_α and if at least one of the series

$$\sum_{\nu=1}^{\infty} \frac{b_2\, b_4 \cdots b_{2\nu}}{b_3\, b_5 \cdots b_{2\nu+1}},\quad \sum_{\nu=2}^{\infty} \frac{b_3\, b_5 \cdots b_{2\nu-1}}{b_4\, b_6 \cdots b_{2\nu}}$$

diverges, then ξ converges to some complex number b.

ParabolicConvergenceTheorem3

Let P_α be a certain parabola in the complex plane with focus (0, 0) going through $z = -1/4$ characterized by the fact that $b_n \in P_\alpha$ if and only if $|b_n| - \mathrm{Re}\left(b_n\, e^{-2\,i\,\alpha}\right) \leq \cos^2(\alpha)\big/2$, $\alpha \in (-\pi/2, \pi/2)$. Let ξ be a continued fraction of the form $\xi = [0; b_1, b_2, \ldots]$. If for all $n = 1, 2, \ldots$, $b_n \in K$ where K is a closed region contained in the interior of P_α and if there exists a real number $M \geq 0$ for which $|b_n| < M$ for all n, then ξ converges to some complex number b.

ParabolicConvergenceTheorem4

Let P_α be a certain parabola in the complex plane with focus (0, 0) going through $z = -1/4$ characterized by the fact that $b_n \in P_\alpha$ if and only if $|b_n| - \mathrm{Re}\left(b_n\, e^{-2\,i\,\alpha}\right) \leq \cos^2(\alpha)\big/2$, $\alpha \in (-\pi/2, \pi/2)$, and for a positive number $d \leq 1/2$, define $C_0(\alpha, d)$, $C_1(\alpha, d)$ to be regions in the complex plane so that $z = x + i\, y \in C_0\,(\alpha, d)$ if and only if

$$x \tan\alpha - d \leq y \leq x \tan\alpha + d$$

and $z \in C_1\,(\alpha, d)$ if and only if

$$x \tan(\alpha) - (1 - d) \leq y \leq x \tan(\alpha) + (1 - d).$$

Let ξ be a continued fraction of the form $\xi = [0; b_1, b_2, \ldots]$. If for all $n = 1, 2, \ldots$, $b_n \in P_\alpha$ and if at least one of the series

$$\sum_{\nu=1}^{\infty} \frac{b_2\, b_4 \cdots b_{2\,\nu}}{b_3\, b_5 \cdots b_{2\,\nu+1}},\ \sum_{\nu=2}^{\infty} \frac{b_3\, b_5 \cdots b_{2\,\nu-1}}{b_4\, b_6 \cdots b_{2\,\nu}}$$

diverges, then ξ converges to some complex number b provided that for all $n = 1, 2, \ldots$, $b_{2\,n}$ lies in one of the regions $C_0\,(\alpha, d)$, $C_1\,(\alpha, d)$ and $b_{2\,n-1}$ lies in the other.

ParabolicConvergenceTheorem5

Let $g_1, g_2, \ldots$ be a sequence of constants with $0 < g_n < 1$ for all n, let $\alpha \in (-\pi/2, \pi/2)$, and let M, ϵ be constants with $\epsilon < 1/2$. Then the continued fraction $\xi = \mathrm{K}(b_n/1) = [0; b_1, b_2, \ldots]$ with elements of the form

$$b_n = e^{2\,i\,\alpha}\, g_n (1 - g_{n+1}) \cos^2(\alpha)\, (u_n + i\, v_n),\ v_n^2 \leq 4\, u_n + 4,$$

converges to a complex number b provided that $|b_n| < M$, $\epsilon < g_n < 1 - \epsilon$, and

$$\sum_{k=1}^{\infty} \prod_{\nu=1}^{k} \left(\frac{1}{g_{\nu+1} - 1}\right)$$

diverges.

ParabolicConvergenceTheorem6

Let $-\pi/2 < \alpha < \pi/2$ and let $\xi = \mathrm{K}(b_n/1) = [0; b_1, b_2, \ldots]$ be a continued fraction whose elements satisfy $|b_n| - \mathrm{Re}\left(b_n\, e^{-2 i \alpha}\right) \leq \cos^2(\alpha)\big/2$ for $n = 1, 2, \ldots$. If there exists a real number $M > 0$ for which $|b_n| < M$ for all n, then ξ converges. Moreover, if the partial quotients b_n are functions of any number of variables, the convergence of ξ to a complex-valued function $b(z)$ is uniform provided that the ranges of the functions $b_n(z)$ satisfy the aforementioned criteria.

ParabolicConvergenceTheorem7

Let $-\pi/2 < \alpha < \pi/2$ and let $\xi = \mathrm{K}(b_n/1) = [0; b_1, b_2, \ldots]$ be a continued fraction whose elements satisfy $|b_n| - \mathrm{Re}\left(b_n\, e^{-2 i \alpha}\right) \leq \cos^2(\alpha)\big/2$ for $n = 1, 2, \ldots$. If the sum

$$\sum_{n=2}^{\infty} \frac{1}{|b_n|\, n}$$

diverges, then ξ converges to a complex number b.

ParabolicConvergenceTheorem8

Let $-\pi/2 < \alpha < \pi/2$ and let $P_{\alpha,n}$ be a sequence of parabolas characterized by the fact that $b_n \in P_{\alpha,n}$ if and only if, for $n = 1, 2, \ldots$,

$$|b_n| - \mathrm{Re}\left(b_n\, e^{-2 i \alpha}\right) \leq \frac{2 n^2}{4 n^2 - 1} \cos^2(\alpha).$$

If $\xi = \mathrm{K}(b_n/1) = [0; b_1, b_2, \ldots]$ is a continued fraction with $b_n \in P_{\alpha,n}$ for all n, then ξ converges to a complex number b provided that the sum

$$\sum_{n=1}^{\infty} \frac{1}{|b_n|\, n \ln(n)}$$

diverges.

ParametricCurveTrace

Given a parametrized curve $\gamma : (a, b) \to \mathbb{R}^2$, the trace of γ is the image set in $\mathbb{R}^2$ which is generated by γ over a given interval. For such a curve γ, its trace is sometimes denoted $\{\gamma\}$.

PartialDenominatorsFromApproximationCoefficientsRecursion

Let ξ be the the regular continued fraction

$$\xi = b_0 + \underset{j=1}{\overset{M}{\mathrm{K}}} \frac{1}{b_j}$$

with $M \leq \infty$, convergents A_n / B_n and approximation coefficients

$$\theta_n = B_n^2 \left| \xi - \frac{A_n}{B_n} \right|.$$

Then the partial denominators b_n can be recovered from the approximation coefficients through

$$b_{n+1} = \left\lfloor \frac{\sqrt{1 - 4\, \theta_{n-1}\, \theta_n} + 1}{2\, \theta_n} \right\rfloor.$$

PellEquationSolution

The Pell equation $x^2 - d\, y^2 = 1$ for nonnegative integers x, y, and d, $\sqrt{d} \notin \mathbb{Z}$ has infinitely many solutions. Let A_n, B_n be the numerators and denominators of the convergents of

$$\sqrt{d} = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

and λ be the length of the period. Then the solutions of the Pell equation are given by

$x_n = A_{n\,k-1}$

$y_n = B_{n\,k-1}$,

where $n \in \mathbb{Z}^+$ for even k and $n/2 \in \mathbb{Z}^+$ for odd k.

PellLikeEquationSolution

Let d be a squarefree integer, c be an integer where $|c| < \sqrt{|d|}$, x and y are integers,

$$r = \frac{x}{y}$$

be a rational number,

$$z = \sqrt{d}$$

be a quadratic irrational, ξ be the regular continued fraction of z, A_n be the convergent numerator of ξ, and B_n be the convergent denominator of ξ. Given

$$\gcd(x,\ y) = 1$$

and

$$x^2 - d\,y^2 = c$$

then

$$\exists_n\,(x = A_n \bigwedge y = B_n).$$

Period1ContinuedFractions

Let d be a squarefree integer,

$$x = \sqrt{d}$$

be a quadratic irrational, ξ be the regular continued fraction of x, l be the regular continued fraction period of ξ, and t be an integer. Given $l = 1$, it follows that

$$\exists_t\, d = 1 + t^2.$$

Period2ContinuedFractions

Let d be a squarefree integer,

$$x = \sqrt{d}$$

be a quadratic irrational, ξ be the regular continued fraction of x, l be the regular continued fraction period of ξ, k be an integer, and X be a natural number. Given $l = 2$, it follows that

$$\exists_{k,X}\left(d = 2\,k + k^2\,X^2 \bigvee d = k + k^2\,X^2\right).$$

Period3ContinuedFractions

Let d be a squarefree integer,

$$x = \sqrt{d}$$

be a quadratic irrational, ξ be the regular continued fraction of x, l be the regular continued fraction period of ξ, k be an integer, and X be a natural number. Given $l = 3$, it follows that

$$\exists_{k,X}\, d = 1 + k^2 + 2\,k\left(3 + 4\,k^2\right) X + \left(1 + 4\,k^2\right) X^2.$$

Period4ContinuedFractions

Let d be a squarefree integer,

$$x = \sqrt{d}$$

be a quadratic irrational,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction of x, b_n be the partial denominator of ξ, l be the regular continued fraction period of ξ, and m be a integer. Given $l = 4$, it follows that

$$b_2 \bmod 2 = 1 \text{ implies } b_1 \bmod 2 = 1$$

and

$$\exists_m\, 2\,b_0 = b_2\,(-1 - b_1\,b_2) + m\left(2\,b_1 + b_1^2\,b_2\right) \text{ and } d = b_0^2 - b_2^2 + m\,(1 + b_1\,b_2).$$

PeriodicContinuedFractionCriterionForPolynomialPellEquation

Let $D(t)$ be a complex polynomial that is not a square. Then the existence of

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{a_n(t)}$$

as a regular continued fraction for $\sqrt{D(t)}$, with a constant period h where

$$\deg(a_n(t)) > 0 \wedge a_h = 2\,a_0 \wedge a_i = a_{h-i}$$

is equivalent to the existence of polynomials $X(t)$ and $Y(t)$ of positive degree such that

$$X(t)^2 - D(t)\,Y(t)^2 = 1.$$

PeriodicPointsOfDerrickEidswickContinuedFraction

Let ξ be a generalized continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{-a}{2}.$$

Then the value a is a periodic point continued fraction for n if $a-1$ is a zero of

$$P_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2\,k+1} x^k.$$

PeriodLengthBoundForContinuedFractionsOfSchinzelSleepers

Let A, B, C be integers where

$$A > 0 \bigwedge \left(4 \gcd\left(A^2, B\right)^2\right) \bmod \left(B^2 - A^2\,C\right) = 0$$

and set

$$D(X) = A^2\,X^2 + 2\,B\,X + C$$

be a Schinzel sleeper. Set

$$\hat{A} = \frac{A}{\gcd(A, B)}$$

$$\Delta = B^2 - A^2\,C$$

Define Δ_1, Δ_2, and Δ_4 by

$$|\Delta| = \Delta_1\,\Delta_2^2\,\Delta_4^4,$$

where Δ_1 and Δ_2 are squarefree integers, and set

$$\hat{\Delta} = \Delta_2\,\Delta_4^2.$$

Let ξ be the regular continued fraction of $\sqrt{D(X)}$, and lp be the regular contin' ued fraction period of ξ. Then

$$\mathrm{lp} \le \left(\begin{cases} 3 \left\lfloor \frac{\ln\left(\sqrt{5}\;\hat{A}\,\hat{\Delta}\right)}{\ln(\phi)} \right\rfloor \hat{\Delta} & \hat{\Delta} \bmod 2 = 0 \\ 2 \left\lfloor \frac{\ln\left(\sqrt{5}\;\hat{A}\,\hat{\Delta}\right)}{\ln(\phi)} \right\rfloor \hat{\Delta} & \hat{\Delta} \bmod 2 = 1 \end{cases} \right).$$

PeriodsRegularContinuedFractionsOfConjugateQuadraticIrrationals

Let ξ be an irrational solution of a quadratic equation with rational coefficients. Then the continued fraction expansion of ξ has the form

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{\begin{cases} b_k & \text{for } k < k_0 \\ b_{k_0+(k-k_0) \bmod m} & \text{for } k \geq k_0 \end{cases}}.$$

The conjugate of ξ has a continued fraction expansion

$$\eta = c_0 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{\begin{cases} c_k & \text{for } j < j_0 \\ c_{k_0+(k-k_0) \bmod m} & \text{for } j \geq j_0 \end{cases}},$$

where

$$c_{k_0+(k-k_0) \bmod m} = b_{m-(k_0+(k-k_0) \bmod m)}.$$

PiContinuedFractionIrrational

Let

$$x = \frac{\pi}{4}$$

and

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{\begin{cases} x & \text{for } n = 1 \\ -x^2 & \text{otherwise} \end{cases}}{-1 + 2\, n}$$

be a generalized continued fraction. Then $\xi = 1$ and x is an irrational number.

PincherleTheorem

Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{b_n}$$

be a generalized continued fraction. Then a minimal three-term recurrence solution X_n exists if and only if ξ converges, and, if such a solution X_n exists, $\xi = -X_0 / X_{-1}$.

PippengerContinuedFractionValue

The finite Pippenger continued fraction

$$\xi = 1 + \cfrac{1}{-1 + t_1 \left(1 + \cfrac{1}{-1 + t_2 \left(1 + \cfrac{1}{-1 + t_3 \left(1 + \cfrac{\cdots}{\cfrac{\ddots}{-1 + t_n}}\right)}\right)}\right)}$$

has the value

$$\xi = \frac{\prod_{k=1}^{n} t_k}{\sum_{j=1}^{n} (-1)^{j+n} \prod_{k=1}^{j} t_k}.$$

PolygonalPolesInPadeApproximants

Let

$$D = \hat{\mathbb{C}} - [-1, 1]$$

be a domain, r be a complex rational function, σ be a Borel measure set, $\hat{\sigma}(z)$ be the Markov function of σ,

$$f = r(z) + \hat{\sigma}(z)$$

be a meromorphic function, V be the poles of f in D, v be a pole, μ be the pole multiplicity of v, $\rho(x)$ be a holomorphic function where $\forall_{x \in [-1,1]} \rho(x) \neq 0$, f_n be the Padé approximants diagonal set at ∞, U be a complex neighborhood of v, and V_n be the poles of f_n in U. Then given

$$\forall_{x \in [-1,1]} D(\sigma)(x) = \frac{\rho(x)}{\sqrt{1 - x^2}}$$

$\mu \geq 3$, then there exists N such that for all $n > N$, V_n are simple poles and are asymptotically configured as a regular polygon.

PolylogarithmContinuedFractionValue

A generalized continued fraction for the polylogarithm function on a single-valued branch on $\mathbb{C}\backslash(-\infty, -1/4)$ is given by

$$-\mathrm{Li}_n(-z) = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_{n,k}\, z}{1},$$

where, letting i and j range from 1 to m

$$A(r, n, m) = \det\left(\frac{(-1)^{i+j+r}}{(r+i+j-1)^n}\right)$$

$$A(r, n, 0) = 1$$

$$a_1 = 1$$

$$a_{2m} = -\frac{A(0, m-1, n)\, A(1, m, n)}{A(0, m, n)\, A(1, m-1, n)}$$

$$a_{2m+1} = -\frac{A(0, m+1, n)\, A(1, m-1, n)}{A(0, m, n)\, A(1, m, n)}.$$

PorterConstant

Porter's constant is a constant C_P appearing in asymptotic formulas for the efficiency of the Euclidean algorithm and also related to continued fractions. It can be written in closed form as

$$C_P = \frac{6\ln(2)\left(\pi^2(4\gamma - 2 + \ln(8)) - 24\,\zeta'(2)\right)}{\pi^4} + \frac{1}{2}$$

where $\zeta'(z)$ is the derivative of the Riemann zeta function, or

$$C_P = \frac{6\ln(2)(48\ln(A) - 2 - \ln(2) - 4\ln(\pi))}{\pi^2} - \frac{1}{2}.$$

The constant has numerical value

$C_P = 1.4670780794339754728977984847072299534499033224148 \ldots$.

Knuth has suggested that C_P be called the Lochs-Porter constant in honor of Lochs, who investigated the related constant

$$\frac{3}{4} - \frac{3\ln(2)}{\pi^2}\left(3\ln(2) - \frac{24\,\zeta'(2)}{\pi^2} + 4\gamma - 2\right) - \frac{6\ln(2)}{\pi^2}\left(\frac{6}{\pi^2}\zeta'(2) - \frac{1}{2}\right) = 0.2173242870 \ldots$$

in a significantly earlier but little-known work on contnued fractions.

PositiveAlgebraicNumbersCanBeRepresentedAsPeriodicBranchedFractionsWithNaturalElements

Any positive algebraic number can be represented as a periodic branched fraction with natural elements.

PositiveProportionOfConvergentDenominatorsForConstrainedPartialQuotientsBoundedHausdorffDimension

Let A be a set of natural numbers, C_A be regular continued fractions whose partial quotients $\subset A$, R_A be finite regular continued fractions whose partial quotients $\subset A$, $D_A(N)$ be denominators d of R_A such that $d \leq N$, $f(N) = \sharp D_A(N)$, and H be the Hausdorff dimension. Then given $H(C_A) > \frac{307}{312}$, it follows that $f(N) = O(N)$.

PositiveProportionOfConvergentDenominatorsForContinuedFractionsWithBoundedHausdorffDimension

Let A be a set of natural numbers, C_A be regular continued fractions whose partial quotients $\subset A$, R_A be finite regular continued fractions whose partial quotients $\subset A$, $D_A(N)$ be denominators d of R_A such that $d \leq N$, $f(N) = \sharp D_A(N)$, and H be the Hausdorff dimension. Then given $H(C_A) > 1 + \left(-27 + \sqrt{633}\right)/16$, $f(N) = O(N)$.

PositiveProportionOfConvergentDenominatorsForPartialQuotientDenominatorsBoundedBySeven

Let A be the natural numbers ≤ 7, C_A be regular continued fractions whose partial quotients $\subset A$, R_A be finite regular continued fractions whose partial quotients $\subset A$, $D_A(N)$ be denominators d of R_A such that $d \leq N$, $f(N) = \sharp D_A(N)$, and H be the Hausdorff dimension. Then $f(N) = O(N)$.

PositiveRealFunction

Let f be a map from the right half-plane of $\mathbb{C}$ to itself which maps the real axis onto itself. Then f is said to be positive real if it is single-valued and analytic in the open right half plane and if the real part $\mathrm{Re}(f(z))$ is positive for all z in the open right half plane.

PringsheimContinuedFractionConvergence

Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{1}$$

be a generalized continued fraction and r_n be real numbers. Given $\exists_{r_n} (0 < r_n < 1 \bigwedge |a_i| < (1 - r_{-1+i})\, r_i)$, then ξ converges.

ProbabilityTheoremForVarianceOfContinuedFractionCoefficients

Let ξ_x be the continued fraction representation of an element $x \in (0, 1)$ where ξ_x has the form $\xi_x = \left[0; b_1^{(x)}, b_2^{(x)}, \ldots\right]$. Then, for fixed K, the set of all x in $(0, 1)$ for which the average of the first K coefficients $b_1^{(x)}, b_2^{(x)}, \ldots, B_K^{(x)}$ differs from $\log_2(K)$ by more than a prescribed value $\epsilon > 0$ is a set of measure zero as $K \to \infty$. Symbolically, for arbtirary $\epsilon > 0$ and for $x \in (0, 1)$ a uniformly distributed random variable,

$$\lim_{K \to \infty} \Pr_{x \in (0,1)} \left\{ \left| \frac{\sum_{n=1}^{\infty} b_n^{(x)} / K}{\log_2(K)} \right| - 1 > \epsilon \right\} = 0.$$

Here $\Pr_{x \in A} \{f(x)\}$ denotes the probability over all random variables x in A that the statement $f(x)$ holds. Moreover, this result cannot be strengthened to say that $\left(\sum_{n=1}^{\infty} a_n / K\right) / \log_2(K) \to 1$ for almost all x in $(0, 1)$.

ProductToContinuedFraction

Let $c_k \neq 0$ for all integer $k \geq 0$ and

$$\xi = \prod_{k=0}^{N} (1 + c_k).$$

Then the continued fraction

$$\eta = 1 + c_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{\begin{cases} (1 + c_0)\, c_1 & \text{for } k = 1 \\ -(1 + c_{k-1}) \frac{c_k}{c_{k-1}} & \text{for } k > 1 \end{cases}}{\begin{cases} 1 & \text{for } k = 1 \\ 1 + (1 + c_{k-1}) \frac{c_k}{c_{k-1}} & \text{for } k > 1 \end{cases}}$$

has the property that for all integers $m \geq 0$ the following identities hold:

$$\prod_{k=0}^{m} (1 + c_k) = 1 + c_0 + \underset{k=1}{\overset{m}{\mathrm{K}}} \frac{\begin{cases} (1 + c_0)\, c_1 & \text{for } k = 1 \\ -(1 + c_{k-1}) \frac{c_k}{c_{k-1}} & \text{for } k > 1 \end{cases}}{\begin{cases} 1 & \text{for } k = 1 \\ 1 + (1 + c_{k-1}) \frac{c_k}{c_{k-1}} & \text{for } k > 1. \end{cases}}$$

ProperlyEquivalent

Two complex numbers $\xi, \eta \in \mathbb{C}$ are called properly equivalent if there exists a properly equivalent unimodular map m with $\eta = m(\xi)$.

ProperlyUnimodularMap

A unimodular map m is called properly unimodular if $\det(m) \in \{\pm 1\}$.

PropertiesOfDiscrepancy

Let $E \subset [0, 1, \omega = \{x_n\}_{n=1}^N$ a sequence of real numbers and define $A(E; N; \omega)$ so that

$A(E; N; \omega) = \sharp\{n : 1 \leq n \leq N \text{ and } \operatorname{frac}(x_n) \in E\}$,

where $\sharp A$ denotes the number of elements of A for all sets A and $\operatorname{frac}(y)$ denotes the fractional part of the element y for all y.

Llet D_N be the discrepancy associated to finite segments of ω, i.e.,

$$D_N(\omega) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{A([\alpha, \beta); N; \omega)}{N} - (\beta - \alpha) \right|.$$

Then necessarily $1/N \leq D_N \leq 1$ where $D_N = 1/N$ if and only if $\{x_n\}_{n=1}^N = \{(n-1)/N\}_{n=1}^N$.

PropertiesOfStarDiscrepancy

Let $E \subset [0, 1, \omega = \{x_n\}_{n=1}^N$ a sequence of real numbers and define $A(E; N; \omega)$ so that

$A(E; N; \omega) = \sharp\{n : 1 \leq n \leq N \text{ and } \operatorname{frac}(x_n) \in E\}$,

where $\sharp A$ denotes the number of elements of A for all sets A and $\operatorname{frac}(y)$ denotes the fractional part of the element y for all y.

For an arbitrary sequence $\omega = \{x_n\}_{n=1}^N$ of real numbers with fractional parts $\operatorname{frac}(x_1), \operatorname{frac}(x_2), \ldots, \operatorname{frac}(x_N)$ ordered increasingly by magnitude, $D_N^* \leq D_N \leq 2\, D_N^*$ and $1/(2\,N) \leq D_N^* \leq 1$. Here D_N and D_N^* denote the discrepancy and star discrepancy, respectively, associated with the finite segments of ω and are defined to be

$$D_N(\omega) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{A([\alpha, \beta); N; \omega)}{N} - (\beta - \alpha) \right|$$

and

$$D_N^* = \max_{i=1,2,\ldots,N} \max\left\{ \left| \frac{i}{N - (x_i)} \right|, \left| \frac{i-1}{N - (x_i)} \right| \right\},$$

respectively. Moreover, the equality $D_N^* = 1/(2\,N)$ holds if and only if $x_n = (2\,n-1)/2\,N$ for $n = 1, 2, \ldots, N$.

Property:ConvergeGenerally

The generalized continued fraction $\xi = \mathrm{K}(a_n / b_n)$ converges generally to $f \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ precisely when its associated M bius transformation $S_n(\xi) = S_n$ converges generally. Here, S_n is defined for all $w \in \mathbb{C}$ by the approximant function

$$S_n(w) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_n}{b_n + w}}}}$$

and is said to converge generally to a constant $\gamma \in \hat{\mathbb{C}}$ if and only if there exists a sequence $\{w_n^\dagger\}$ from $\hat{\mathbb{C}}$ such that $\lim_{n\to\infty} S_n(w_n) = \gamma$ whenever

$$\lim_{n\to\infty} \inf_{k\geq n} m\left(w_k, w_k^\dagger\right) > 0$$

where m denotes Ahlfors '"chordal metric." One can easily show that conver-gence in the general sense is an immediate consequence of convergence in the classical sense.

Property:LyapunovExponentExists

Let $G(x)$ denote the Gauss map which is defined piecewise as

$$G(x) = \begin{cases} x & \text{for } x = 0 \\ x - \lfloor x \rfloor & \text{for } x \neq 0, \end{cases}$$

and let $\lambda(\gamma)$ denote the values of the Lyapunov exponents (if they exist) of G for $\gamma \in \mathbb{R}$ an arbitrary real number. By the ergodicity of G, one can conclude that the Lyapunov exponent exists for the orbits under G of almost all (with respect to either Lebesgue or Gauss measure) $\gamma \in \mathbb{R}$. Moreover, the value $\lambda(\gamma)$ can be computed explicitly for elements $\gamma \in \mathbb{R}$ whose G orbits do omit well-defined Lyapunov exponents and is precisely

$$\lambda(\gamma) = -\frac{2}{\ln(2)} \int_0^1 \frac{\ln(x)}{1+x} \, d\ell(x) = \frac{\pi^2}{6\ln(2)},$$

where ℓ denotes the Lebesgue measure on $\mathbb{R}$. Despite this, the collection N of initial points $\gamma \in \mathbb{R}$ for which $\lambda(\gamma)$ fails to exist is actually dense in $\mathbb{R}$ as, for example, $\mathbb{Q} \subset N$.

PurelyPeriodicSequence

A sequence $a_1, a_2, a_3, \ldots$ is purely periodic if there exists a positive integer $p \in \mathbb{Z}^+$ such that $a_{n+p} = a_n$ for every positive integer $n \in \mathbb{Z}^+$.

QuadraticIrrationalsAreBadlyApproximableNumbers

Let α be a quadratic irrational number where $0 \leq \alpha \leq 1$ and ξ be the regular continued fraction of α. Then ξ is badly approximable.

QuadraticIrrationalsWithPeriodTwelve

Let d be a natural number where

$d \bmod 4 = 3$

and ξ be the regular continued fraction of $\sqrt{d}$, $l(d)$ be the period of ξ, and $S(X)$ be natural numbers where $d \leq X$ and $l(d) = 12$. Then

$\left| S(X) \right| = O\left(\sqrt{X} \ln(X)\right)$.

QueffelecTheorem

The continued fraction of a Thue-Morse sequence is transcendental.

QuinticBoundOnComputingTimeOfContinuedFractionsMethodForPolynomialRealRootIsolation

Let A be a continued fraction method with root bounds algorithm, p be the input polynomial of A, n be the polynomial degree of p, and $t(A)$ be the computing time set of A. Then there exists a constant $c > 0$ such that $\forall\, n\, \exists\, p$ such that $t(A) \geq c\, n^5$.

RadiusOfConvergenceForGSeriesAssociatedToRogersRamanujanContinuedFraction

Let τ be an irrational number, define the modular nome by

$$q = e^{2 i \pi \tau}$$

as the parameter of the Rogers Ramanujan continued fraction,

$$G_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2} z^k}{(q;\, q)_m}$$

be its associated holomorphic function, and R_q be the holomorphic radius set of $G_q(z)$. Then

$$R_q = \liminf_{n \to \infty} |1 - q^n|^{1/n}.$$

RamanujanSelfReciprocalContinuedFraction

The continued fraction

$$\xi(x) = 1 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{k^2}{x}$$

with the closed form value

$$\xi(x) = \frac{1}{2}\left(\psi^{(0)}\left(\frac{x+3}{4}\right) - \psi^{(0)}\left(\frac{x+1}{4}\right)\right)$$

for $\mathrm{Re}(x) > -1$ fulfills the self-reciprocal identity

$$\xi(x) = \int_0^{\infty} \xi(s) \sin\left(\frac{x \pi s}{2}\right) ds.$$

RationalsInTheFareyProcess

Every rational number p/q in lowest terms with $0 < p/q < 1$ appears at some stage of the Farey process provided that the process begins with the numbers $0/1$ and $1/1$.

ReddmannTheorem

Given a real number ξ with finite regular continued fraction expansion

$$\xi = 0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}.$$

and finite base-b expansion $(0.\, d_1\, d_2 \cdots d_N{}_b$, the terms of the two expansions are equal ($b_n = d_n$ for $n = 1, 2, \ldots, N$) for $N \leq 2$ when and only when:

a) For $N = 1$, $\xi = 1/b_1$ and $b = b_1^2$.

b) For $N = 2$, ($\xi = 4/9$ and $b = 6$) or ($\xi = \frac{2\,100\,332}{13\,051\,463\,049}$ and $b = 38\,614\,134$).

RegularChain

A regular chain is an infinite product $T_0\, T_1 \cdots T_n \cdots$ where $T_0 = V_1^{b_0}$, $b_0 \in \mathbb{Z}$, $T_1 \neq V_1$, and

$$\begin{cases} T_n \in \{V_j,\, E_j,\, C\} & \text{for } \det(T_0\, T_1 \cdots T_{n-1}) = \pm 1 \\ T_n \in \{V_j,\, C\} & \text{for } \det(T_0\, T_1 \cdots T_{n-1}) = \pm i \end{cases}$$

for $n \geq 1$ such that no $n_0 \in \mathbb{Z}^+$, $j \in \{1, 2, 3\}$ exist for which $T_n = V_j$ for all $n \geq n_0$.

The matrices used here are defined as follows:

$$V_1 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix},\ V_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix},\ V_3 = \begin{pmatrix} 1-i & i \\ -i & i+1 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1-i & i \end{pmatrix},\ E_2 = \begin{pmatrix} 1 & i-1 \\ 0 & i \end{pmatrix},\ E_3 = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & i-1 \\ 1-i & i \end{pmatrix}.$$

RegularContinuedFraction

A continued fraction ξ is said to be regular if it has the form

$$\xi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots}}},$$

where $b_k \in \mathbb{Z}$ for all $k = 0, 1, 2, \ldots$ and where $b_k > 0$ for $k \geq 1$. The regular fraction ξ above can also be written $\xi = [b_0; b_1, b_2, \ldots]$ or, using Gauss notation,

$$\xi = b_0 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_m}.$$

The terms b_k are said to be both the partial quotients and the partial denominators of ξ, as the partial numerators of ξ are all identically 1.

It is not uncommon in literature for the unmodified term "continued fraction" to mean "regular continued fraction," and despite an apparent loss of generality in doing so, no such loss exists. Indeed, a well-known result in the study of continued fractions is the existence of an equivalence transformation $r = \{r_m\}$ between any generalized continued fraction ξ and an associated regular continued fraction ξ_{reg}, whereby it follows that any theory for generalized continued fractions holds for regular fractions and vice versa. Regular continued fractions are especially useful when representing irrationals, for example, because the convergents of regular continued fractions are the so-called best rational approximations thereof.

RegularContinuedFractionApproximationsSpecialFractions

Let $0 < \xi < 1$ be an irrational number with regular continued fraction representation

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Let $a \in \mathbb{R}$, $a > 1$. Then for all ξ with infinitely many $b_k > a$, there exist infinitely many rational numbers p/q, such that the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{\sqrt{a^2 + 4}} \frac{1}{q}$$

has infinitely many solutions for p/q.

RegularContinuedFractionAsymptoticDenominatorDistribution

For almost every $x \in [0, 1]$ with associated regular continued fraction $\xi(x) = [0; b_1^x, b_2^x, \ldots]$, the digit j appears in the expansion of ξ with density

$$\frac{2\ln(1+j) - \ln(j) - \ln(2+j)}{\ln 2}.$$

Said a different way, for any $i \in \mathbb{Z}^+$,

$$\lim_{n\to\infty} \frac{\text{card}\{\kappa : b_\kappa = i,\ 1 \le \kappa \le n\}}{n} = \frac{2\ln(1+j) - \ln(j) - \ln(2+j)}{\ln(2)} = \frac{1}{\ln(2)} \ln\left(1 + \frac{1}{i(i+2)}\right)$$

for almost all $x \in [0, 1]$ where here, $[0; b_1, b_2, \ldots]$ is the regular continued fraction expansion associated to x. This result was originally discovered by Lévy in the early 20th century.

RegularContinuedFractionAveragePartialQuotientGrowth

Let $0 < \xi < 1$ be an irrational number with regular continued fraction representation

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

For almost all ξ the following identity holds:

$$\liminf_{n\to\infty} \frac{1}{n} \ln(\ln(n)) \left(\max_{1\le j\le n} b_j\right) = \frac{1}{\ln(2)}.$$

RegularContinuedFraction:CommonNotations

Common notations for the regular continued fraction

$$\xi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cfrac{1}{\dots}}}}$$

include

$$\xi = [b_0; b_1, b_2, b_3, \dots]$$

$$\xi = \langle b_0; b_1, b_2, b_3, \dots \rangle$$

$$\xi = b_0 + \frac{1}{b_1 +} \; \frac{1}{b_2 +} \; \frac{1}{b_3 +} \dots$$

$$\xi = b_0 + \frac{1 \rfloor}{\lceil b_1} + \frac{1 \rfloor}{\lceil b_2} + \frac{1 \rfloor}{\lceil b_3} + \dots \quad \text{(Pringsheim)}$$

and

$$\xi = b_0 + \operatorname*{K}_{k=1}^{\infty} \frac{1}{b_k} \text{ (Gauss).}$$

In Gauss's notation, the uppercase K stands for "Kettenbruch," which is German for "continued fraction."

While most authors use a_k instead of b_k to denote the terms of a regular continued fraction, the b_k convention is followed here since it is consistent with notations for generalized continued fractions in which a_k denotes a partial numerator and b_k a partial denominator.

Common notations for the nth convergent of a continued fraction include p_n/q_n and A_n/B_n, the former being more prevalent in older papers and the latter being more common in the recent literature. Here, the notation A_n/B_n is used.

RegularContinuedFraction:CompleteQuotient

Given a regular continued fraction ξ of the form

$$\xi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots}}},$$

the nth complete quotient ζ_n of ξ is the continued fraction obtained by ignoring the first n partial denominators $b_0, \dots, b_{n-1}$, i.e.,

$$\zeta_n = b_n + \cfrac{1}{b_{n+1} + \cfrac{1}{b_{n+2} + \cfrac{1}{\ddots}}}.$$

Other notations for ζ_n are $\zeta_n = [b_n; b_{n+1}, b_{n+2}, \dots]$ or, in Gauss notation,

$$\zeta_n = b_n + \operatorname*{K}_{m=n+1}^{\infty} \frac{1}{b_m}.$$

RegularContinuedFraction:CompleteQuotientDenominator

Let ζ_n be the nth complete quotient of a regular continued fraction $\xi = [b_0; b_1, b_2, \ldots]$, i.e., ζ_n is the regular continued subfraction of the form

$$\zeta_n = b_n + \cfrac{1}{b_{n+1} + \cfrac{1}{b_{n+2} + \cfrac{1}{\ddots}}}.$$

The denominators of ζ_n are the positive integers $b_n, b_{n+1}, b_{n+2}, \ldots$ which, more generally, can be described as the collection of elements b_k for $k \geq n$.

RegularContinuedFraction:CompleteQuotientNumerator

Let ζ_n be the nth complete quotient of a regular continued fraction $\xi = [b_0; b_1, b_2, \ldots]$, i.e., ζ_n is the regular continued subfraction of the form

$$\zeta_n = b_n + \cfrac{1}{b_{n+1} + \cfrac{1}{b_{n+2} + \cfrac{1}{\ddots}}}.$$

Due to the fact that ζ_n is regular, the numerators of ζ_n are all identically 1. Said another way, the continued fraction ζ_n can be written in Gauss notation as

$$\zeta_n = b_n + \underset{m=n+1}{\overset{\infty}{\mathrm{K}}} \frac{a_m}{b_m}$$

where, for all $m = n+1, n+2, \ldots$, $a_m \equiv 1$ are its numerators.

RegularContinuedFraction:Convergence

A regular continued fraction ξ of the form

$$\xi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

with nth convergent $\xi_n = [b_0; b_1, b_2, \ldots, b_n]$ is said to converge to a value x if $\xi_n \to x$ as $n \to \infty$. Note that the concept of regular continued fraction convergence is merely an example of generalized continued fraction convergence where the continued fractions in question have partial numerators a_k satisfying $a_k = 1$, $k = 1, 2, 3, \ldots$.

RegularContinuedFraction:Convergent

Given a regular continued fraction ξ of the form

$$\xi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}},$$

its nth convergent ξ_n is the finite continued fraction obtained by truncating ξ at the nth level, i.e.,

$$\xi_n = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}}.$$

Alternate notations for ξ_n include the shorthand $\xi_n = [b_0; b_1, b_2, \ldots, b_n]$, as well as Gauss notation

$$\xi_n = b_n + \underset{m=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_m}.$$

Note that this definition is nothing more than a specialized version of the definition of convergent for a generalized continued fraction except that the fraction ξ in question has partial numerators a_k which satisfy $a_k = 1$, $k = 1, 2, 3, \ldots$.

RegularContinuedFraction:ConvergentDenominator

Given a regular continued fraction ξ of the form

$$\xi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}},$$

its nth convergent denominator B_n is the expression in the denominator of the nth convergent $\xi_n = A_n / B_n$ where ξ_n is the finite continued subfraction of the form

$$\xi_n = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}}.$$

Note that this definition is nothing more than a specialized version of the definition given for a generalized continued fraction except that the fraction ξ in question has partial numerators a_k which satisfy $a_k = 1$, $k = 1, 2, 3, \ldots$.

RegularContinuedFraction:ConvergentNumerator

Given a continued fraction ξ of the form

$$\xi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}},$$

its nth convergent numerator A_n is the expression in the numerator of the nth convergent $\xi_n = A_n / B_n$ where ξ_n is the finite continued subfraction of the form

$$\xi_n = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}}.$$

Note that this definition is nothing more than a specialized version of the definition given for a generalized continued fraction except that the fraction ξ in question has partial numerators a_k which satisfy $a_k = 1$, $k = 1, 2, 3, \ldots$.

RegularContinuedFractionConvergentsApproximationProperty

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}$$

be a regular continued fraction with $b_k \in \mathbb{Z}^+$ and A_k / B_k the sequence of its convergents. Then

$$\left| \xi - \frac{A_n}{B_n} \right| \leq \frac{1}{B_n \, B_{n+1}} \leq \frac{1}{B_n^2}.$$

RegularContinuedFractionConvergentsApproximationsBetterThanRoot5

For any continued fraction ξ

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n, set

$$\lambda_n = \frac{1}{B_n^2} \frac{1}{\left| \xi - \frac{A_n}{B_n} \right|}.$$

Then for all $c > \sqrt{5}$ there is ξ with finitely many $\lambda_n > c$.

RegularContinuedFractionConvergentsIrreducibility

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}$$

be a regular continued fraction with $b_k \in \mathbb{Z}^+$ and p_k / q_k the sequence of its convergents.

Then for all $n \in \mathbb{Z}^+$, the following identities hold for the convergents:

$$\gcd(p_n, q_n) = 1$$

$$\gcd(p_n, q_{n+1}) = 1$$

$$\gcd(p_{n+1}, q_n) = 1.$$

RegularContinuedFractionConvergentsMembership

Let p/q be an irreducible fraction. Let ξ be a positive real number. If

$$\left|\xi - \frac{p}{q}\right| \leq \frac{1}{2\, q^2}$$

or

$$\left|p^2 - q^2\, \xi^2\right| \leq \xi.$$

Then p/q is a convergent of the regular continued fraction of ξ.

RegularContinuedFraction:Divergence

Divergence of a regular continued fraction ξ of the form

$$\xi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

with nth convergent $\xi_n = [b_0; b_1, b_2, \ldots, b_n]$ occurs when ξ_n fails to converge to a finite expression as $n \to \infty$. Note that this definition is nothing more than a specialized version of the definition given for a generalized continued fraction except that the fraction ξ in question has partial numerators a_k which satisfy $a_k = 1$, $k = 1, 2, 3, \ldots$.

RegularContinuedFraction:Expansion

Given a constant c, a regular continued fraction expansion is an expression of the form

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with partial denominators b_k taken from some domain, usually positive integers, such that $\xi = c$.

RegularContinuedFraction:FiniteContinuedFraction

A finite regular continued fraction ξ is a regular continued fraction of the form

$$\xi = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_n}}}}$$

which terminates after only finitely many terms.

A well-known result in the theory of continued fractions is that the associated continued fraction $\xi(\alpha)$ of an element $\alpha \in \mathbb{R}$ is finite (and hence is of the form $\xi(\alpha) = [\beta_0; \beta_1, \beta_2, \ldots, \beta_n]$, $\beta_k \in \mathbb{Z}$ for all k, $\beta_n \neq 0$ for $n \geq 1$) precisely when $\alpha \in \mathbb{Q}$. For that reason, finite continued fractions play an important role in many branches of mathematics due to the fact that irrationals (i.e., elements whose associated continued fractions are infinite) can be estimated arbitrarily well by such terms.

RegularContinuedFractionFirstThreeConsecutiveConvergentsApproximationPropertyForPartialQuotientsGreaterThanOne

For any continued fraction ξ

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n, set

$$\lambda_n = \frac{1}{B_n^2} \frac{1}{\left| \xi - \frac{A_n}{B_n} \right|}.$$

Then $b_{n+2} \geq 2$ implies that $\max(\lambda_n, \lambda_{n+1}, \lambda_{n+2}) > 2\sqrt{2}$.

RegularContinuedFractionFiveConsecutiveConvergentsApp

roximationPropertyForPartialQuotientsOneTwo

For any continued fraction ξ

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n, set

$$\lambda_n = \frac{1}{B_n^2} \frac{1}{\left|\xi - \frac{A_n}{B_n}\right|}.$$

Then $b_{n+1} = 1$ and $b_{n+2} = 2$ implies

$\max(\lambda_n, \lambda_{n+1}, \lambda_{n+2}, \lambda_{n+3}, \lambda_{n+4}) > \left(2 + 5 \sqrt{10}\right) / 6.$

RegularContinuedFractionFordCircleChains

Let ξ be a positive real number with regular continued fraction expansion

$$\xi = b_0 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

and convergents A_n / B_n.
Then the Ford circles of the convergents A_n / B_n form a chain, meaning the Ford circle of the convergent A_k / B_k is tangent to the Ford circle of the convergent A_{k+1} / B_{k+1}.

RegularContinuedFractionGeneralConvergentsApproximationProperty

Let x be an irrational number and

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction of x with convergents A_n / B_n. If

$$\left|\xi - \frac{A_n}{B_n}\right| \geq \frac{1}{\sqrt{r^2 + 4}\; B_n^2}$$

holds for all $n \in \{m - 1, m, m + 1\}$, the inequality $b_{m+1} < r$ holds.

RegularContinuedFractionHalfRegularContinuedFractionConvergentsRelation

Let ξ have the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

and A_n / B_n the sequence of its convergents. Let ξ have the half-regular continued fraction expansion

$$\xi = \beta_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{\varepsilon_k}{\beta_k}$$

and p_n / q_n the sequence of its convergents with $\varepsilon_k \in \{-1, 1\}$, $\beta_k \in \mathbb{Z}^+$, $\beta_k \geq 2$ and $\beta_k + \varepsilon_{k+1} \geq 2$, $\varepsilon_1 = \operatorname{sgn}(\xi)$, $|\beta_1 - 1/|\xi|| < 1/2$.

Then for all $n \geq 0$ there exists a unique function $k(n)$, such that

$$\frac{A_{n+1}}{B_{n+1}} = \frac{p_{k(n)+1}}{q_{k(n)+1}} \text{ or } \frac{A_{n+1}}{B_{n+1}} = \frac{p_{k(n)+2}}{q_{k(n)+2}}$$

with the latter case if and only if $b_{k(n)+2} = 1$. For almost all ξ

$$\lim_{n \to \infty} \frac{k(n)}{n} = \frac{\ln(2)}{\ln(\phi)}$$

holds.

RegularContinuedFractionLevelSetFact1

Let $\mathbb{I}$ be the set of irrational numbers from the interval $[0, 1]$. Let $\xi \in \mathbb{I}$ have the regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n. Let

$$\mathcal{F}_\alpha = \left\{ x \in \mathbb{I} : \lim_{n \to \infty} \frac{\prod_{j=1}^{n} b_j}{B_n} = \alpha \right\};$$

then

$\mathcal{F}_\alpha = \emptyset$ if $\alpha \notin [0, 1]$.

RegularContinuedFractionLevelSetFact2

Let $\mathbb{I}$ be the set of irrational numbers from the interval [0, 1]. Let $\xi \in \mathbb{I}$ have the regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n. Let

$$\mathcal{F}_\alpha = \left\{ x \in \mathbb{I} : \lim_{n \to \infty} \frac{\prod_{j=1}^{n} b_j}{B_n} = \alpha \right\}$$

and

$$\mathcal{N} = \{x \in \mathbb{I} : \exists_n \forall_{m>n}\, b_m = 1\},$$

then

$$\mathcal{N} \subset \mathcal{F}_0.$$

RegularContinuedFractionLevelSetFact3

Let $\mathbb{I}$ be the set of irrational numbers from the interval [0, 1]. Let $\xi \in \mathbb{I}$ have the regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n and η be the quadratic surd

$$\eta_k = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{k}.$$

Let

$$\mathcal{F}_\alpha = \left\{ x \in \mathbb{I} : \lim_{n \to \infty} \frac{\prod_{j=1}^{n} b_j}{B_n} = \alpha \right\}$$

then

$$\eta_k \in \mathcal{F}_{-\ln(k) / \left(\ln\left(k/2 + \left(k^2/4 + 1\right)^{1/2}\right)\right)}$$

and

$$\lim_{k \to \infty} -\ln(k) / \left(\ln\left(k/2 + \left(k^2/4 + 1\right)^{1/2}\right)\right) = 1.$$

RegularContinuedFractionLevelSetFact4

Let $\mathbb{I}$ be the set of irrational numbers from the interval [0, 1]. Let $\xi \in \mathbb{I}$ have the regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n. Let

$$\mathcal{F}_\alpha = \left\{ x \in \mathbb{I} : \lim_{n\to\infty} \frac{\prod_{j=1}^{n} b_j}{B_n} = \alpha \right\}$$

and

$$\mathcal{N} = \left\{ x \in \mathbb{I} : \lim_{j\to\infty} b_j = \infty \right\}$$

then

$$\mathcal{N} \subset \mathcal{F}_1.$$

RegularContinuedFractionLevelSetFact5

Let $\mathbb{I}$ be the set of irrational numbers from the interval [0, 1]. Let $\xi \in \mathbb{I}$ have the regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n. Let

$$\mathcal{F}_\alpha = \left\{ x \in \mathbb{I} : \lim_{n\to\infty} \frac{\prod_{j=1}^{n} b_j}{B_n} = \alpha \right\}.$$

Then for almost all $x \in \mathbb{I}$,

$$x \in \mathcal{F}_{12 \ln(2) \ln(K) / \pi^2}$$

holds.

RegularContinuedFractionLevelSetFact6

Let $\mathbb{I}$ be the set of irrational numbers from the interval [0, 1]. Let $\xi \in \mathbb{I}$ have the regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n/B_n. Let

$$\mathcal{F}_\alpha^* = \begin{cases} \left\{x \in \mathbb{I} : \limsup_{n\to\infty} \frac{\prod_{j=1}^n b_j}{B_n} \ge \alpha\right\} & \text{for } \alpha \ge 12 \ln(2) \ln(K) / \pi^2 \\ \left\{x \in \mathbb{I} : \limsup_{n\to\infty} \frac{\prod_{j=1}^n b_j}{B_n} \le \alpha\right\} & \text{for } \alpha \le 12 \ln(2) \ln(K) / \pi^2. \end{cases}$$

Then for $\alpha_q = 1 - 1/\left(q^2 \ln(q)\right)$

$$\left\{x \in \mathbb{I} : \forall_{j \ge 1}\, b_j \ge q\right\} \subset \mathcal{F}_{\alpha_q}^*.$$

RegularContinuedFractionLevelSetFact7

Let $\mathbb{I}$ be the set of irrational numbers from the interval [0, 1]. Let $\xi \in \mathbb{I}$ have the regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n/B_n. Let

$$\mathcal{F}_\alpha = \left\{x \in \mathbb{I} : \lim_{n\to\infty} \frac{\prod_{j=1}^n b_j}{B_n} = \alpha\right\}.$$

Then if $x \in \mathcal{F}_1$, then

$$\lim_{j\to\infty} b_j = \infty.$$

RegularContinuedFractionLevelSetFact8

Let $\mathbb{I}$ be the set of irrational numbers from the interval [0, 1]. Let $\xi \in \mathbb{I}$ have the regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents $A_n(\xi)/B_n(\xi)$. Let

$$\mathcal{F}_\alpha = \left\{ x \in \mathbb{I} : \lim_{n\to\infty} \frac{\prod_{j=1}^{n} b_j}{B_n} = \alpha \right\}.$$

Then the Hausdorff dimension of $\mathcal{F}_\alpha$

$$\dim(\mathcal{F}_\alpha) = f(\alpha),$$

where

$$f(\alpha) = \max\left(-\hat{t}(\alpha),\, 0\right).$$

Here, $\hat{t}(\alpha)$ is the Legendre transform of $t(\alpha)$

$$\hat{t}(\alpha) = \sup_{c\in\mathbb{R}}(c\,\alpha - t(c))$$

and $t(\beta)$ is defined implicitly through $P(t(\beta),\, \beta) = 0$ and

$$P(t,\, \beta) = \lim_{n\to\infty} \frac{1}{n} \ln\left(\sum_{b_1=1}^{\infty} \cdots \sum_{b_n=1}^{\infty} B_n\left(\underset{k=1}{\overset{n}{\mathrm{K}}} \frac{1}{b_k} \right)^{-2\,t} \prod_{j=1}^{n} b_j^{-2\,\beta} \right).$$

The function $f(\alpha)$ is strictly convex in [0, 1] and continuous and real-analytic in (0, 1). Its maximal value is

$$f\left(\frac{12\ln(2)\ln(K)}{\pi^2} \right) = 1.$$

Furthermore,

$$f(0) = 0$$

$$f(1) = \frac{1}{2}$$

$$\lim_{\alpha\to 0^+} f'(\alpha) = \infty$$

$$\lim_{\alpha\to 1^-} f'(\alpha) = -\infty.$$

RegularContinuedFractionLevelSetFact9

Let $\mathbb{I}$ be the set of irrational numbers from the interval [0, 1]. Let $\xi \in \mathbb{I}$ have the regular continued fraction expansion

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n. Let

$$\mathcal{F}_\alpha = \left\{ x \in \mathbb{I} : \lim_{n\to\infty} \frac{\prod_{j=1}^{n} b_j}{B_n} = \alpha \right\}$$

and

$$\mathcal{I}_q = \left\{ x \in \mathbb{I} : x = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k} \bigwedge \forall_{j \geq 1}\, b_j \geq q \right\}.$$

Then the Hausdorff dimension $\dim_H$ of $\mathcal{I}_q$

$$\dim_H \mathcal{I}_q \sim \frac{1}{2} + \frac{1}{2} \frac{\ln(\ln(q))}{\ln(q)}$$

as $q \to \infty$.

RegularContinuedFractionMeanConvergentsApproximationProperty

For any continued fraction ξ

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n, set

$$\lambda_n = \frac{1}{B_n^2} \frac{1}{\left| \xi - \frac{A_n}{B_n} \right|}.$$

Then

$$\liminf_{m} \frac{1}{m} \sum_{i=0}^{m-1} \lambda_i \geq \sqrt{5}\,.$$

RegularContinuedFractionNConsecutiveConvergentsApproximationProperty

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}$$

be a regular continued fraction with $b_k \in \mathbb{Z}^+$ and A_k / B_k the sequence of its convergents.

Then for all $n \in \mathbb{Z}^+$, $k \in \mathbb{Z}^+$

$$\min\left(\left\{B_n^2 \left|\xi - \frac{A_n}{B_n}\right|, B_{n+1}^2 \left|\xi - \frac{A_{n+1}}{B_{n+1}}\right|, \ldots, B_{n+k}^2 \left|\xi - \frac{A_{n+k}}{B_{n+k}}\right|\right\}\right) < c_k$$

where

$$c_k = \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} \left(\frac{3-\sqrt{5}}{2}\right)^{2k+3}.$$

The constant c_k is the best possible constant.

RegularContinuedFraction:PartialDenominator

The partial denominators of a regular continued fraction ξ of the form

$$\xi = b_0 + \underset{m=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_m}$$

(where N may be infinite) are the elements b_k, $k = 0, 1, 2, \ldots$.

RegularContinuedFraction:PartialNumerator

Given a collection of integers $\{b_k\}_{k=0}^{\infty}$ with $b_n \neq 0$ for $n \geq 1$, a regular continued fraction ξ is a (finite or infinite) fraction of the form

$$\xi = b_0 + \underset{m=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_m},$$

i.e., a fraction whose partial numerators a_k satisfy $a_k = 1$ for all $k = 1, 2, \ldots, N$ (where here, N may be infinite). Therefore, by definition, the partial numerators of an arbitrary regular continued fraction ξ are all identically 1.

RegularContinuedFraction:Period

A regular continued fraction ξ of the form

$$\xi = b_0 + \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_m}$$

is said to be periodic provided its terms eventually repeat from some point forward, and the minimal number of repeating terms in such a fraction is called its period. Said differently, if $\xi = [b_0; b_1, b_2, \ldots]$ is a regular continued fraction and if k is the smallest positive integer for which $b_{r\,k+m} = b_m$ for all $m = 1, 2, \ldots, k$, $r = 0, 1, 2, \ldots$, then ξ is said to be periodic and k is said to be the period of ξ.

Given the continued fraction ξ above with nth convergent $\xi_n = A_n / B_n$, it can be shown that ξ is generated by successive recursive composition of the linear fractional transformation $s = s(w)$, where

$$s(w) = \frac{A_{k-1}\, w + A_k}{B_{k-1}\, w + B_k}.$$

By studying transformations of this form— specifically the fixed points of such transformations— several key continued fraction convergence results can be derived. Such techniques can be found throughout the works of Abel, Lane, Stolz, Pringsheim, Perron, Schwerdtfeger, and Wall.

RegularContinuedFractionReciprocal

Given the regular continued fraction expansion of a real number ξ

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}$$

(for N possibly ∞), the reciprocal continued fraction when $b_0 = 0$ is

$$\frac{1}{\xi} = b_1 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_{k+1}}$$

and the reciprocal continued fraction for $b_0 > 0$ is

$$\frac{1}{\xi} = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_{k-1}}.$$

RegularContinuedFractionSecondThreeConsecutiveConvergentsApproximationPropertyForPartialQuotientsGreaterThanOne

For any continued fraction ξ

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n, set

$$\lambda_n = \frac{1}{B_n^2} \frac{1}{\left|\xi - \frac{A_n}{B_n}\right|}.$$

Then $b_{n+2} \geq 2$ implies that $\lambda_{n+1} > 5/2 \bigvee \max(\lambda_n, \lambda_{n+2}) > 5/2$.

RegularContinuedFractionsOfSquareRootsOfRationals

Let $\xi > 1$ be a rational number and $\sqrt{\xi} \notin \mathbb{Z}$. Then the regular continued fraction expansion of ξ

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

is periodic with period ν and the periodic part consists of a symmetric initial sequence followed by the term $2\, b_0$.

For $k \geq 1$ the following relations hold:

$$b_{(k \bmod \nu)+1} = b_{k+1}$$

$$b_\nu = 2\, b_0$$

$b_{\nu-k} = b_k$ for $1 \leq k < \nu - 1$.

RegularContinuedFraction:StrictVanVleckFraction

Let ξ be a regular continued fraction of the form

$$\xi = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

where each partial denominator b_k is an arbitrary complex number and let $w_n = [0; b_1, b_2, \ldots, b_n]$ denote the nth convergent of ξ. Suppose further that $\mathrm{Re}\,(b_n) > 0$ for all n and that, for $\theta < \pi/2$ arbitrary, $|\arg(b_n)| < \theta$. Such a fraction ξ is said to be a strict Van Vleck fraction with angle θ.

RegularContinuedFractionSumAndProductOfTwoConsecutiveConvergentsApproximationProperty

For any continued fraction ξ

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with convergents A_n / B_n, set

$$\lambda_n = \frac{1}{B_n^2} \frac{1}{\left|\xi - \frac{A_n}{B_n}\right|}.$$

Then $\lambda_n \lambda_{n+1} > \lambda_n + \lambda_{n+1} > \max\left((\lambda_n - 1)\lambda_n^2, (\lambda_{n+1} - 1)\lambda_{n+1}^2\right) > 4$.

RegularContinuedFractionsWithIdenticalTails

Let ξ and η be two irrational numbers with regular continued fraction expansions

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

$$\eta = c_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{c_k}.$$

If and only if there exist integers a, b, c, and d with $a\,d - b\,c = 1$, then there exist integers N, M, such that for all $n \geq N$

$$b_n = c_{M+n}.$$

RegularContinuedFractionThreeConsecutiveConvergentsApproximationProperty

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}$$

be a regular continued fraction with $b_k \in \mathbb{Z}^+$ and A_n / B_n the sequence of its convergents. Then for all $n \in \mathbb{Z}^+$, either

$$\left|\xi - \frac{A_n}{B_n}\right| < \frac{1}{\sqrt{5}\, B_n^2}$$

or

$$\left|\xi - \frac{A_{n+1}}{B_{n+1}}\right| < \frac{1}{\sqrt{5}\, B_{n+1}^2}$$

or

$$\left|\xi - \frac{A_{n+2}}{B_{n+2}}\right| < \frac{1}{\sqrt{5}\, B_{n+2}^2}.$$

RegularContinuedFractionTwoConsecutiveConvergentsApproximationProperty

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k}$$

be a regular continued fraction with $b_k \in \mathbb{Z}^+$ and A_k / B_k the sequence of its convergents. Then for all $n \in \mathbb{Z}^+$, either

$$\left|\xi - \frac{A_n}{B_n}\right| < \frac{1}{2\, B_n^2}$$

or

$$\left|\xi - \frac{A_{n+1}}{B_{n+1}}\right| < \frac{1}{2\, B_{n+1}^2}.$$

RegularContinuedFraction:VanVleckFraction

Let ξ be a regular continued fraction of the form

$$\xi = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots}}}$$

where each partial denominator b_k is an arbitrary complex number and let $w_n = [0; b_1, b_2, \ldots, b_n]$ denote the nth convergent of ξ. Suppose further that $\operatorname{Re}(b_n) > 0$ for all n. Then ξ is called a Van Vleck fraction.

RegularContinuedFractionWithAveragePartialQuotientGrowth

Let $0 < \xi < 1$ be an irrational number with regular continued fraction representation

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Let $k, K, M_0 \in \mathbb{R}^+$ with $k > 1$ and $K \geq 2$. Let the b_k fulfill the conditions

$$\max_{\ln^2(k)\, M\, K^{k^M} < n \leq \ln^2(k)\, (M+1)\, K^{k^M}} a_n = \left\lfloor K^{k^{M+1}} \right\rfloor$$

$$\max_{\ln^2(k)\, (M+1)\, K^{k^M} < n \leq \ln^2(k)\, (M+1)\, K^{k^{M+1}}} a_n = \left\lfloor K^{k^{M+1}} \right\rfloor$$

where $M \in \mathbb{Z}^+$ and $M > M_0$. Then

$$\liminf_{n \to \infty} \frac{1}{n} \ln(\ln(n)) \max_{1 \leq j \leq n} b_j = \frac{1}{\ln(2)}.$$

RegularContinuedFractionWithPartialDenominatorRestrictionTheoremHirst1

Let $\{\phi_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of natural numbers and let α be chosen so that $\sum_{n=1}^{\infty} \phi_n^{-\alpha}$ converges for real positive α. Let A have the property that

$$\sum_{n=1}^{\infty} \frac{\theta(\phi_n - A)}{\phi_n^{\alpha}} \leq \frac{1}{2^{\alpha/2}}.$$

Then the set of all continued fractions

$$\left\{ \xi : \xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k} \bigwedge b_k \geq A \bigwedge b_k \in \{\phi_n\} \right\}$$

has Hausdorff dimensions less than or equal to $1/2$.

RegularContinuedFractionWithPartialDenominatorRestrictionTheoremHirst2

Let $\{\phi_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of natural numbers and let α be chosen so that $\sum_{n=1}^{\infty} \phi_n^{-\alpha}$ diverges for real positive α.

Then the set of all continued fractions

$$\left\{\xi \,:\, \xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k} \bigwedge b_k \in \{\phi_n\}\right\}$$

has Hausdorff dimensions less than or equal to $\alpha/2$.

RegularContinuedFractionWithPartialDenominatorRestrictionTheoremHirst3

Let $\{\phi_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of natural numbers and let α be chosen so that $\sum_{n=1}^{\infty} \phi_n^{-\alpha}$ diverges for real positive α.

Then the set of all continued fractions

$$\left\{\xi \,:\, \xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k} \bigwedge b_k \in \left\{n^b\right\}_{n=1}^{\infty} \bigwedge n \in \mathbb{Z}^+ \bigwedge b_k \geq k^b\right\}$$

has Hausdorff dimensions less than or equal to $b/2$.

RegularExpansionUnderSchinzelCondition

Let A,B,C be integers where $B > 0$, $C > 0$,

$\left|B^2 - A^2\,C\right| = 1$

and $\gcd\left(A^2, 2\,B, C\right)$ is squarefree,

$d = \gcd\left(A^2, 2\,B, C\right)$,

N be a natural number, X be a formal variable,

$$y = \frac{B}{A}$$

be a rational number,

$$\eta = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{a_n}$$

be the regular continued fraction of y, k be the length of the continued fraction η,

$D(X) = A^2\,X^2 + 2\,B\,X + C$

be a Schinzel sleeper,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction of $\sqrt{D(X)}$, and p be the regular continued fraction period of ξ. Given d is squarefree, then

$p = 1 + k$

and

$\exists_N\,\forall_{X>N}\,(b_0 = A\,X + a_0 \bigwedge \forall_{1\leq n\leq k}\,b_n = a_n \bigwedge b_{1+k} = 2\,(A\,X + a_0))$.

RegulatorOfRealQuadraticFieldUsingContinuedFractions

Let D be a square-free positive integer and for the regular continued fraction for $\sqrt{D}$

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

$$b_{p+n} = b_n$$

define

$$P_0 = 0$$

$$Q_0 = 1$$

$$P_{n+1} = b_n\, Q_n - P_n$$

$$Q_{n+1} = \frac{D - P_{n+1}^2}{Q_n}$$

$$\theta_n = \frac{\sqrt{D} + P_n}{Q_n}.$$

Then the fundamental unit for $\mathbb{Q}\left(\sqrt{D}\right)$ is

$$\epsilon = \begin{cases} \frac{1}{2}\left(A_{r-1} + \sqrt{D}\, B_{r-1}\right) & \text{if there is } r < \left\lfloor \frac{p}{2} \right\rfloor,\ Q_r = 4 \\ A(p-1) + \sqrt{D}\, B(p-1) & \text{otherwise} \end{cases}$$

$$= \begin{cases} 2 \prod_{i=0}^{r} \theta(i) & \text{if there is } r < \left\lfloor \frac{p}{2} \right\rfloor,\ Q_r = 4 \\ \left(\sqrt{D} + P\left(\frac{p+1}{2}\right)\right) \prod_{i=0}^{\lfloor p/2 \rfloor} \theta(i) & \text{if } p \text{ is odd} \\ Q\left(\frac{p}{2}\right) \left(\prod_{i=0}^{p/2} \theta(i)\right) & \text{if } p \text{ is even.} \end{cases}$$

RegulatorOfRealQuadraticFieldUsingNearestIntegerContinuedFractions

Let D be a square-free positive integer and for the regular continued fraction for $\sqrt{D}$

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

$$b_{p+n} = b_n$$

define

$$P_0 = 0$$

$$Q_0 = 1$$

$$P_{n+1} = b_n Q_n - P_n$$

$$Q_{n+1} = \frac{D - P_{n+1}^2}{Q_n}.$$

If $D \neq 5$, $D \neq 13$, and there are x and y so that $\left|x^2 - y^2 D\right| = 4$, then there is an $r \leq \lfloor p/2 \rfloor$ with $Q_r = 4$ and the fundamental unit for $\mathbb{Q}\left(\sqrt{D}\right)$ is

$$\epsilon = A_{r-1} + \sqrt{D}\, B_{r-1}.$$

ReinerTheorem

Let K be a division ring and let $R = K[x$ be the ring of polynomials in an indeterminate x with coefficients in K, where it is assumed that x commutes with all elements of K. For $f_1, f_2, \ldots, f_N \in R$, define A and B as the formal numerator and denominators of the continued fraction having terms f_i and denote this as $\xi = [f_1, f_2, \ldots, f_N] \sim A/B$, where A/B can be defined by the relation

$$\begin{pmatrix} f_1 & 1 \\ 1 & 0 \end{pmatrix} . \begin{pmatrix} f_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} f_N & 1 \\ 1 & 0 \end{pmatrix} . \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}.$$

Let $f \to f^*$ denote any homomorphism of $(R, +)$ into itself, which leaves K elementwise fixed and satisfies $(a\, f)^* = a\, f^*$ for all $a \in K$, $f \in R$. Then $\xi = [f_1, f_2, \ldots, f_N] \sim A/B$ and $A, B \in K$ implies $\xi^* = [f_1^*, f_2^*, \ldots, f_N^*] \sim A/B$.

RemarkOnDivergenceOfCertainJFractions

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z+b_1 - \cfrac{a_2}{z+b_2 - \cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$, and

$$\sum_{j=1}^{\infty}\left(\left|\frac{a_j - 1}{4}\right| + \left|b_j\right|\right) < \infty.$$

Furthermore, A_n / B_n denote the $n\,t\,h$ convergence of f, let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$, and for notational convenience, let $u_j = 2\, b_j$ for $j \geq 0$ and let $v_j = 1 - 4\, a_j$ for $j \geq 1$. If $x = \cos(\vartheta)$, $\omega = e^{\pm i\,\vartheta}$ for $0 < \vartheta < \pi$, then

$$\left|\frac{A_{n+1}(x)}{B_{n+1}(x)} - \frac{A_n(x)}{B_n(x)}\right| \geq 2\,|1 + w|^2\, K^{-2} \prod_{j=1}^{n}\left|1 - v_j\right|$$

for

$$K = K(\omega) = 2\left(1 + \sum_{r=1}^{\infty}|1 - w|^{-r}\, \rho_{-1}(0)\, \rho_0(1) \cdots \rho_{r-2}(1)\right),$$

$$\rho_k(R) = \sum_{j=k+1}^{\infty}\left(\left|u_j\right| R^{1/2}\left(1 + R^{j-k}\right) + \left|v_j\right|\left(R + R^{j-k}\right)\right).$$

In particular, this shows that for all $x \in (-1, 1)$ the continued fraction $f(x)$ diverges.

RemarkOnGeneralAnalyticContinuedFractionsAndBranchPoints

Let $F(z)$ be a general analytic limit periodic continued fraction of the form

$$F(z) = \cfrac{1}{\lambda(z) + b_0(z) - \cfrac{a_1(z)}{\lambda(z)+b_1(z) - \cfrac{a_2(z)}{\lambda(z)+b_2(z) - \cfrac{a_3(z)}{\ddots}}}}$$

where $a_n(z) \not\equiv 0$, $b_{n-1}(z)$ and $\lambda(z)$ are holomorphic functions of z in a region $G \subset \mathbb{C}$ for $n \geq 1$, and where $\lim_{n\to\infty} a_n(z) = 1/4$, $\lim_{n\to\infty} b_n(z) = 0$ hold uniformly on each compact subset of G. If the open set G^* is defined so that $G^* = G \backslash S$ where $S = \{z \in G : \lambda(z) \in [-1, 1]\}$, if $\hat{\omega}(z)$ is defined on each component of G^* so that $\hat{\omega}(z) = \omega(\lambda(z))$ where $\omega(z) = z - \left(z^2 - 1\right)^{1/2}$ with roots chosen positive for $z > 1$, $z \in \mathbb{C} \backslash [-1, 1]$, and if G^{**} is defined to be the 2-sheeted Riemannian surface of $\hat{\omega}(z)$ over G obtained by analytic extension of $\hat{\omega}$ from each compo∴ nent of G^* across S into a second copy of G, then the point $z_0 \in S$ is a branch point of $\hat{\omega}(z)$ extended onto G^{**} if and only if $\lambda(z_0) = \pm 1$ of odd order.

RemarkOnGeneratingFunctionsAndJFractionConvergence

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z+b_1 - \cfrac{a_2}{z+b_2 - \cfrac{a_3}{\ddots}}}}$$

where a_n, $b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, where $\lim a_n = 1/4$ and $\lim b_n = 0$ hold, and where $A_n(z)/B_n(z)$ denotes the nth approximant of f. Suppose that

$$\sum_{j=1}^{\infty}\left(\left|\frac{a_j - 1}{4}\right| + \left|b_j\right|\right) < \infty$$

and, in addition, let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$ where, for convenience, the notation $u_j = 2\, b_j$ for $j \geq 0$ and let $v_j = 1 - 4\, a_j$ for $j \geq 1$ is adopted. For $|\omega| \leq 1$, $\omega \neq 1$, $|z| < 1$, define the function $G_k(z)$ to be the generating function of the sequence $S_k^{(n)}(z)$ for $n > k$, i.e., $G_k(z) = \sum_{n=k+1}^{\infty} z^n\, S_k^{(n)}(z)$ where

$$S_k^{(n)}(\omega) = 1 - w^{n-k} + \sum_{r=1}^{n-k-1} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega)\left(1 - w^{n-j_r}\right),$$

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\, u_j\left(1 - w^{j-k}\right) + w\, v_j\left(1 - w^{j-k-1}\right)\right),$$

with $c_{k,j}(\pm 1) = \pm(j-k)\, u_j + (j-k-1)\, v_j$ by definition. It follows, then, that

$$G_k(z) = \frac{z^{k+1}(1-w)}{(1-z)(1-z\,w)}\left(1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega)\, z^{j_r-k}\right),$$

and hence that:

1. $G_k(z)$ converges absolutely for $|\omega| \leq 1$, $\omega \neq 1$, $|z| \leq 1$.

2. Absolute convergence in $G_k(z)$ also happens provided that $\sum_{j=k+1}^{\infty}\left|c_{k,j}(\omega)\, z^j\right| < \infty$, a criterion satisfied whenever $|z| < 1$, $|\omega| \leq 1$, and u_j, v_j, $c_{k,j}$ are bounded for $j > k \geq -1$.

3. The function $G_k(z)$ satisfies

$$\lim_{z \to 1}(1-z)\, G_k(z) = S_k(z)$$

where

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega).$$

RepresentabilityOfRealNumbersAsSumsOfNumbersWhose

FractionalPartsContinuedFractionsPartialQuotientsAreNotLessThan2

Every real number x can be represented as a sum of two numbers whose regular continued fractions $x = (a_1 + \xi_1) + (a_2 + \xi_2)$

$$\xi_j = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

with $2 \leq b_k$ for all k and $i = 1, 2$.

RepresentationTheoremForAleksenkoSpectrum

Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be a regular continued fraction with convergents A_n / B_n, f_n be a sequence where $\left| \xi - \frac{A_{f(n)}}{B_{f(n)}} \right| < \frac{1}{2 B_{f(n)}^2}$,

$Q_n = B_{f(n)}$,

$\mu(\alpha)(t)$ be a Minkowski diagonal function,

$m(\alpha) = \limsup_{t \to \infty} t\, \mu(\alpha)(t)$,

α_ν be the complete quotients of ξ,

$(\alpha_\nu)^* =$ from the continued fraction $[0; b_\nu, \ldots, b_1]$,

$$\xi^{-1} = \frac{1}{\xi}$$

be its regular continued fraction, α_ν^{-1} be the complete quotient continued fraction of ξ^{-1},

$$F(x, y) = \frac{(1 - x\, y)^2}{4\,(1 - x)\,(1 - y)\,(x\, y + 1)}$$

$$G(x, y) = \frac{1}{4}\,(x + y + 1)$$

$$m_n(\alpha) = \begin{cases} G\left(\alpha_\nu^*, \alpha_{\nu+2}^{-1}\right) & \exists_\nu \{Q_n, Q_{1+n}\} = \{B_{-1+\nu}, B_{1+\nu}\} \\ F\left(\alpha_{\nu+1}^*, \alpha_{\nu+2}^{-1}\right) & \exists_\nu \{Q_n, Q_{1+n}\} = \{B_\nu, B_{1+\nu}\} \end{cases}$$

and then define

$i(\alpha) = \liminf_{n \to \infty} m_n(\alpha)$

$I = \left\{m : \exists_\xi\, i(\alpha) = m\right.$.

Then $\exists_{w_0}\, [1/4, w_0] \subset I$.

RepresentationTheoremForMinkowskiSpectrum

Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{a_n}$$

be a regular continued fraction with convergents A_n / B_n, f_n be a sequence where $\left|\xi - \frac{A_{f(n)}}{B_{f(n)}}\right| < \frac{1}{2\,B_{f(n)}^2}$,

$Q_n = B_{f(n)}$,

$\mu(\alpha)(t)$ be a Minkowski diagonal function set,

$m(\alpha) = \limsup_{t\to\infty} t\,\mu(\alpha)(t)$, and

define M to be real numbers m where

$\exists_\xi\, m(\alpha) = m$.

Then $M \subset [1/4, 1/2]$ and $\{1/4, 1/2\} \in M$.

RestrictedDenominatorContinuedFractions

Let F_k be the set of all infinite regular continued fractions with partial denominators between 1 and k.

$$F_k = \left\{\xi : \xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j} \bigwedge b_j \in \mathbb{Z}^+ \bigwedge 1 \le b_j \le k\right\}.$$

Let P_k be the closed interval

$$P_k = \left[\underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{\begin{cases} k & \text{for } j \bmod 2 = 1 \\ 1 & \text{for } j \bmod 2 = 0 \end{cases}},\ \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{\begin{cases} 1 & \text{for } j \bmod 2 = 1 \\ k & \text{for } j \bmod 2 = 0 \end{cases}}\right].$$

Let O_k be the set

$$O_k = \bigcup_{m=1}^{\infty} \left[\underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{\begin{cases} b_j & \text{for } 1 \le j \le m \\ k & \text{for } (j-m) \bmod 2 = 1 \\ 1 & \text{for } (j-m) \bmod 2 = 0 \end{cases}},\ \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{\begin{cases} b_j & \text{for } 1 \le j \le m \\ 1 & \text{for } (j-m) \bmod 2 = 1 \\ k & \text{for } (j-m) \bmod 2 = 0 \end{cases}}\right]$$

where $b_j \in \mathbb{Z}^+$ and $1 \le b_j \le k$ and $b_m \ne m$. Then F_k is the following set-theoretic difference: $F_k = P_k \setminus O_k$.

ReversePeriodicRegularContinuedFraction

Let $\xi > 1$ be an irrational solution of a quadratic equation with rational coefficients of the form

$$\xi = \frac{P + \sqrt{D}}{Q}$$

with $p,\ Q,\ D \in \mathbb{Z}$ with $P \geq 0$, $D > 0$, and $Q > 0$, and $Q \mid (D - P^2)$. Let the regular continued fraction expansion of ξ be purely periodic

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_{k \bmod m}}.$$

If the conjugate of ξ is

$$\eta = \frac{P - \sqrt{D}}{Q}$$

then the following expansion holds:

$$-\frac{1}{\eta} = b_m + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_{m-k \bmod m}}.$$

RichardsFareyProcessApproximationTheorem

Given any irrational number ξ with $0 < \xi < 1$, the Farey process (zeroed in on ξ) gives a sequence of best left and best right approximations to ξ. Furthermore, every best left/right approximation arises in this way.

RichardsFareyProcessRealNumberTheorem

Every rational number p/q in lowest terms with $0 < p/q < 1$ appears at some stage of the Farey process.

RichardsFastContinuedFractionAlgorithmTheorem

Given any irrational number ξ with $0 < \xi < 1$, the fast continued fraction algorihtm gives precisely the set of ultra-close approximations to ξ.

RogersRamanujanContinuedFractionConvergenceAtRootsOfUnity

Let τ be a complex number, define the modular nome by
$q = e^{2 i \pi \tau}$,
let $r(\tau)$ be the Rogers Ramanujan continued fraction of q, and $x = a/b$ be a rational number. Then $r(\tau)$ converges $\Longleftrightarrow b \bmod 5 \neq 0$ and

$$b \bmod 5 \neq 0 \Rightarrow r\left(\frac{a}{b}\right) = \frac{a \left(e^{2 \pi i a b/5}\, r(0)\right)^{5/b}}{b}.$$

RogersRamanujanContinuedFractionExpressibleAsRadicals

Let τ be a complex number, define the modular nome by
$q = e^{2 i \pi \tau}$,
let $r(\tau)$ be the Rogers Ramanujan continued fraction of q,
$j(\tau) = J(\tau)$
the Klein invariant J, and $f(\tau)$ be the dehomogenized icosahedral equation.
Then $j(\tau)$ is expressible as radicals $\wedge$ $f(\tau)$ is reducible $\Longleftrightarrow$ $r(\tau)$ is expressible as radicals.

ScaledApproximationCoefficientsLimit

Let $0 < \xi < 1$ be an irrational number with regular continued fraction representation

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

be a continued fraction and A_n / B_n the sequence of its convergents.
Then the following identity hold for almost all ξ:

$$\lim_{n \to \infty} \frac{1}{n} \ln \left| \xi - \frac{A_n}{B_n} \right| = -\frac{\pi^2}{6 \ln(2)}.$$

SchmidtExpansionConsecutiveConvergents

Let ξ be a complex number with $\text{Im}(\xi) \geq 0$ with Schmidt expansion $\xi = M_1 \cdot M_2 \cdot \ldots \cdot M_N$ and convergents $\left\{A_n^{(0)} / B_n^{(0)},\ A_n^{(1)} / B_n^{(1)},\ A_n^{(\infty)} / B_n^{(\infty)}\right\}$.

Then for all $A_n^{(l)} / B_n^{(l)}$, $l \in \{0, 1, \infty\}$, if $P_1\left(\tau^j(\xi, 1, \mathbb{I}_2)\right) \in \mathbb{C}$ the following holds:

$\left|A_{n+1}^{(l)}\right| \geq \left|A_n^{(l)}\right|$ and $\left|B_{n+1}^{(l)}\right| \geq \left|B_n^{(l)}\right|$.

SchmidtExpansionConvergents

Let ξ be a complex number with $\text{Im}(\xi) \geq 0$ with Schmidt expansion convergents $\left\{A_n^{(0)} / B_n^{(0)},\ A_n^{(1)} / B_n^{(1)},\ A_n^{(\infty)} / B_n^{(\infty)}\right\}$. Then for almost all $A_n^{(l)} / B_n^{(l)}$, $l \in \{0, 1, \infty\}$, the following hold:

$$\lim_{n \to \infty} \frac{\left|\ln\left(B_n^{(l)}\right)\right|}{n} = C$$

$$\lim_{n \to \infty} \frac{1}{n} \left|\ln\left(\xi - \frac{A_n^{(l)}}{B_n^{(l)}}\right)\right| = -\frac{2\,C}{\pi}.$$

SchmidtExpansionMultipleConvergents

Let ξ be a complex number with $\text{Im}(\xi) \geq 0$ with Schmidt expansion $\xi = M_1 \cdot M_2 \cdot \ldots \cdot M_N$ and convergents $\left\{p_n^{(0)} / q_n^{(0)},\ p_n^{(1)} / q_n^{(1)},\ p_n^{(\infty)} / q_n^{(\infty)}\right\}$. Then for all $p_n^{(l)} / q_n^{(l)}$, $l \in \{0, 1, \infty\}$, the following holds:

If $M_j \in \{V_1, E_2, E_3\}$:

$$\left(p_{n+1}^{(\infty)} = p_n^{(\infty)} \bigvee p_{n+1}^{(\infty)} = i\, p_n^{(\infty)}\right) \bigwedge \left(q_{n+1}^{(\infty)} = q_n^{(\infty)} \bigvee q_{n+1}^{(\infty)} = i\, q_n^{(\infty)}\right);$$

If $M_j \in \{V_2, E_3, E_1\}$:

$$\left(p_{n+1}^{(0)} = p_n^{(0)} \bigvee p_{n+1}^{(0)} = i\, p_n^{(0)}\right) \bigwedge \left(q_{n+1}^{(0)} = q_n^{(0)} \bigvee q_{n+1}^{(0)} = i\, q_n^{(0)}\right).$$

If $M_j \in \{V_3, E_1, E_2\}$:

$$\left(p_{n+1}^{(1)} = p_n^{(1)} \bigvee p_{n+1}^{(1)} = i\, p_n^{(1)}\right) \bigwedge \left(q_{n+1}^{(1)} = q_n^{(1)} \bigvee q_{n+1}^{(1)} = i\, q_n^{(1)}\right).$$

ScottWallCaseOfLeightonConjecture

Let ξ be a C-fraction,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n z^{\alpha_n}}{1},$$

m be a natural number, D be the unit disk, and B be the domain boundary set of D. Then given

$a_n = a$

$\alpha_n = m^n$

it follows that ξ converges in D to a meromorphic function and that B is the natural meromorphic boundary.

SeidelEquivalenceTheorem

Let

$$\xi_1 = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_1(n)}{b_1(n)}$$

be a generalized continued fraction,

$$\xi_2 = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_2(n)}{b_2(n)}$$

be a generalized continued fraction, and r_n be an equivalence transformation. Then

$$\exists_{r_n} (r_0 = 1 \bigwedge r_n \neq 0 \bigwedge b_1(n) = r_n\, b_2(n) \bigwedge a_1(n) = r_{n-1}\, r_n\, a_2(n)) \Longleftrightarrow \xi \text{ and } \eta \text{ are equivalent.}$$

SeidelMultiplicationTheorem

Let

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction with convergents p_k/q_k. Let ρ_k be a sequence with $\rho_k \neq 0$ for all k and let

$$\eta = \rho_0\, b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{\rho_{k-1}\, \rho_k\, a_k}{\rho_k\, b_k}$$

be a continued fraction with convergents P_k/Q_k. Then the following identities hold:

$$\eta = \rho_0\, \xi$$

$$P_k = \rho_0 \times \left(\prod_{j=1}^{k} \rho_j\right) \times p_k$$

$$Q_k = \left(\prod_{j=1}^{k} \rho_j\right) \times q_k.$$

SeidelSternTheorem

A positive continued fraction $\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} 1/b_n$ converges if and only if $\sum_n^\infty b_n = \infty$. If $\sum_n^\infty b_n < \infty$, then ξ diverges generally.

SeidelSternTheoremTransformed

A positive continued fraction $\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} a_n/b_n$ converges if and only if its Stern-Stolz series diverges to ∞, i.e., if and only if $\sum_{n=1}^\infty b_n \prod_{k=1}^n a_k^{(-1)^{k+n+1}} = \infty$.

SemiUniqueRegularChainRepresentationsOfCertainComplexNumbers

For any complex number $\xi \in \mathbb{C} \backslash \mathbb{Q}(i)$ which is properly equivalent to some real number $r \in \mathbb{R}$, there exist precisely two regular chains $c\, h_1\, \xi$ and $c\, h_2\, \xi$ representing ξ.

SeriesToContinuedFraction

Let $c_k \neq 0$ for all integer $k \geq 0$ and

$$\xi = \sum_{k=0}^{N} c_k.$$

Then the continued fraction

$$\eta = c_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{\begin{cases} c_1 & \text{for } k = 1 \\ -\frac{c_k}{c_{k-1}} & \text{for } k > 1 \end{cases}}{\begin{cases} 1 & \text{for } k = 1 \\ 1 + \frac{c_k}{c_{k-1}} & \text{for } k > 1 \end{cases}}$$

has the property that for all integer $m \geq 0$ the following identities hold:

$$\sum_{k=0}^{m} c_k = c_0 + \underset{k=1}{\overset{m}{\mathrm{K}}} \frac{\begin{cases} c_1 & \text{for } k = 1 \\ -\frac{c_k}{c_{k-1}} & \text{for } k > 1 \end{cases}}{\begin{cases} 1 & \text{for } k = 1 \\ 1 + \frac{c_k}{c_{k-1}} & \text{for } k > 1 \end{cases}}.$$

ShiftTransformation

The unmodified term "shift transformation" refers to the mapping of a regular continued fraction $\xi = [b_0; b_1, b_2, \ldots]$ to the translated regular continued fraction $\xi_1 = [b_1; b_2, b_3, \ldots]$. This idea can be generalized to the n-fold composition of the above transformation which takes ξ to the regular continued fraction $\xi_n = [b_n; b_{n+1}, b_{n+2}, \ldots]$. Restricted to numbers x in the interval $(0, 1)$ with corresponding regular continued fractions $\xi(x) = [0; b_1, b_2, \ldots]$, the shift transformation T is defined so that $T : [0; b_1, b_2, \ldots] \mapsto [0; b_2, b_3, \ldots]$ and is given by the closed-form expression

$$T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

The map T is studied by way of measure theory and functional analysis, for example, and in addition to the fact that Gauss' measure is invariant with respect to it, T can also be shown to be ergodic and indecomposable with respect to Lebesgue measure. Results of this variety can be found in Billingsley, among others.

In general, however, there are a number of differing shift transformations which are also studied from a variety of different contexts. For example, Schmidt proved that analogous theorems to the above hold for the analogously-defined shift transformation τ for regular chain and dually regular chain representations of a complex number $z \in \mathbb{C}$. Various other, more specialized types of shift transformations exist as well, for example the β-shift and (a, b)-shift transformations.

SleszynskiPringsheimContinuedFractionValueSet

For every complex number f from the unit disk ($|f| \leq 1$) with the exception $f = 0$, there exists a Śleszyński-Pringsheim continued fraction

$$\xi = b_0 + \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{a_k}{b_k}$$

such that $\xi = f$.

SleszynskiPringsheimTheorem

Let

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a Śleszyński-Pringsheim continued fraction. Then ξ converges absolutely to some value f with $0 < |f| \leq 1$.

SpecialRelativityVelocityAdditionsContinuedFraction

Let k be a nonsquare integer, $k > 5$. Let $x_0 \in \mathbb{Q}$, $x_0 > 0$. Define the sequence x_n through

$$x_{n+1} = \frac{x_n + 1}{1 + x_n / k}.$$

Let

$$\sqrt{k} = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

be the regular continued fraction expansion with convergents p_k / q_k. Then there are at most finitely many solutions of the equation $x_n = p_k / q_k$.

A closed form for x_n is given by

$$x_n = \frac{\sqrt{k}\left(\left(\frac{-1}{\sqrt{k}+k}\right)^n \left(-\sqrt{k} + x_0\right) + \left(\frac{1}{\sqrt{k}-k}\right)^n \left(\sqrt{k} + x_0\right)\right)}{\left(\frac{-1}{\sqrt{k}+k}\right)^n \left(\sqrt{k} - x_0\right) + \left(\frac{1}{\sqrt{k}-k}\right)^n \left(\sqrt{k} + x_0\right)}.$$

SquareProductConjecture

Given sequences $\{a_k\}_{k=1}^{\infty} = \{a_k(z)\}_{k=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty} = \{b_k(z)\}_{k=1}^{\infty}$ of complex-valued functions analytic on domains Ψ and Ω, respectively, for which the infinite continued fraction $\underset{k=1}{\overset{\infty}{\mathrm{K}}}(a_k / b_k)$ converges in $\mathbb{C} \cup \{\infty\}$,

$$\prod_{k=1}^{\ell} \left(\underset{j=k}{\overset{\infty}{\mathrm{K}}} \frac{a_j}{b_j}\right)^2 = B_{\ell-1}^2 \left(\underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k} - \underset{k=1}{\overset{\ell-1}{\mathrm{K}}} \frac{a_k}{b_k}\right)^2.$$

Here, $B_{\ell-1}$ refers the terms of the three-term recurrence relation

$$B_m = b_m B_{m-1} + a_m B_{m-2},$$

$B_{-1} = 0$, $B_0 = 1$, satisfied by the finite convergents of $\underset{k=1}{\overset{\infty}{\mathrm{K}}}(a_k / b_k)$.

StablePolynomialCriteria

Let the complex polynomial of degree n

$$p_n(z) = z_n + \sum_{k=0}^{n-1} a_k z^k$$

be stable (meaning for all roots z_k it holds that $\mathrm{Re}(z_k) < 0$).

The polynomial $p_n(z)$ is stable if and only if

$$t_n(z) = \frac{\left(\sum_{k=0}^{n/2} \mathrm{Re}\left(a_{n-(2k+1)}\right) z^{n-(2k+1)}\right) + i \left(\sum_{k=1}^{n/2} \mathrm{Im}\left(a_{n-(2k)}\right) z^{n-(2k)}\right)}{z^n + i \left(\sum_{k=0}^{n/2} \mathrm{Im}\left(a_{n-(2k+1)}\right) z^{n-(2k+1)}\right) + \left(\sum_{k=1}^{n/2} \mathrm{Re}\left(a_{n-(2k)}\right) z^{n-(2k)}\right)}$$

can be written in the form

$$t_n(z) = \underset{k=1}{\overset{n}{\mathrm{K}}} \frac{1}{i\, t_k + d_k\, z}$$

where $t_k \in \mathbb{R}$ and $d_k > 0$ for all $1 \leq k \leq n$.

StarDiscrepancyBoundsForFunctionsOfBoundedVariation

For a function $f : [0, 1] \to \mathbb{R}$ with bounded variation $V(f)$,

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_0^1 f(t)\, dt \right| \leq D_N^* \, V(f).$$

StarDiscrepancyOfARealSequence

Let $E \subset [0, 1, \omega = \{x_n\}_{n=1}^N$ a sequence of real numbers and define $A(E; N; \omega)$ so that

$A(E; N; \omega) = \sharp \{n \,:\, 1 \leq n \leq N \text{ and } \mathrm{frac}(x_n) \in E\}$,

where $\sharp A$ denotes the number of elements of A for all sets A and $\mathrm{frac}(y)$ denotes the fractional part of the element y for all y.

Given a sequence $\{x_n\}_{n=1}^N$ of real numbers with fractional parts $\mathrm{frac}(x_1)$, $\mathrm{frac}(x_2)$, … , $\mathrm{frac}(x_N)$ ordered increasingly by magnitude, the star discrepancy D_N^* associated with the sequence is defined to be

$$D_N^* = \max_{i=1,2,\ldots,N} \max \left\{ \left| \frac{i}{N - \mathrm{frac}(x_i)} \right|, \left| \frac{i-1}{N - \mathrm{frac}(x_i)} \right| \right\}.$$

SternStolzTheorem

Let

$$\xi = b_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be a regular continued fraction with $b_n \in \mathbb{C}$ and A_n / B_n the sequence of its convergents. Then if $\sum_{n=1}^{\infty} |b_n| < \infty$,

1. the continued fraction ξ diverges generally.

2. the sequences $\{A_{2n+m}\}_n$ and $\{B_{2n+m}\}_n$ converge absolutely to finite values $\mathcal{A}_m$ and $\mathcal{B}_m$, respectively (for $m = 0,\ 1$).

3. $\mathcal{A}_1 \mathcal{B}_0 - \mathcal{A}_0 \mathcal{B}_1 = 1$.

StieltjesMomentProblem

The Stieltjes moment problem, investigated as part of Stieltjes '1894 exposition on continued fractions, seeks to determine necessary and sufficient conditions for a sequence $\{m_n\}$ of real numbers to be of the form

$$m_n = \int_0^\infty x^n \, d\mu(x)$$

for some measure μ defined on [0, ∞. Originating as part of an investigation on relationships between J-fractions, S-fractions, and infinite series, Stieltjes himself gave a necessary and sufficient condition for the existence of a solution. In the decades since, this problem has been extended and analyzed by many authors, resulting in a variety of conditions for existence and uniqueness of solutions thereto.

Stieltjes 'original condition states that a solution $\{m_n\}$ of the moment problem exists if and only if the Hankel determinants satisfy

$$\begin{vmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{vmatrix}, \begin{vmatrix} m_1 & m_2 & \cdots & m_{n+1} \\ m_2 & m_3 & \cdots & m_{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n+1} & m_{n+2} & \cdots & m_{2n+1} \end{vmatrix} > 0$$

for all $n = 0, 1, 2, \ldots$, though this says nothing about whether the solution is unique. Several other criteria quantify the uniqueness of solutions to the Stielt-jes moment problem, e.g. Carleman's condition which states that any solution $\{m_n\}$ will be unique provided that

$$\sum_{n=1}^{\infty} m_n^{-(2n)^{-1}} = \infty.$$

Several other results related to continued fractions can be found in the work of Alkhiezer, e.g., who proves that precisely one solution $\{m_n\}$ to the Stieltjes moment problem exists whenever $\{m_n\}$ is defined in terms related to the ele-ments of an S-fraction $\xi = \xi(z)$ of the form

$$\xi = a_0 + \cfrac{1}{b_1 z + \cfrac{1}{a_1 + \cfrac{1}{b_2 z + \cdots}}}$$

and at least one of the series $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ diverges, $a_0 \geq 0$, $a_k, b_k \in \mathbb{Z}^+$, $k = 1, 2, 3, \ldots$. Additional results concerning the Stieltjes moment problem can be found in the works of Bultheel et al. and van Assche, among others.

StieltjesRogersTheorem

Let $\phi(z) = 1 + \sum_{n\geq 1}^{\infty} \phi_n(z)\, z^n / n!$ be an exponential generating function satisfying a Stieltjes-Rogers addition formula with coefficients w_n. Let $\Phi(z) = 1 + \sum_{n\geq 1}^{\infty} \phi_n(z)\, z^n$ be the generating function corresponding to $\phi(z)$. Then

$$\Phi(z) = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1 - z\, c_n}{z^2\, d_n}$$

where

$$c_n = \varphi(j,\ j+1)(z) - \varphi(j-1,\ j)(z)$$

is a formal power series,

$$d_n = \frac{w_n}{w_{n-1}}$$

is a real number, and

$$\varphi(j,\ k)(z) = k!\, z^k\, \varphi(j)(z).$$

StrongBestRationalApproximation

A fraction p/q is called a strong best rational approximation of the real number ξ if

$$|q\, \xi - p| < |s\, \xi - r|$$

for any integers r and s such that $s \leq q$ and $p/q \neq r/s$.

Every strong best rational approximation p/q is also a best approximation of ξ.

Let ξ have the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{M}{\mathrm{K}}} \frac{1}{b_k}$$

(for M possibly ∞) with convergents A_n/B_n.

Then every convergent A_n/B_n is strong best rational approximation of ξ.

SumOfRegularContinuedFractionPartialDenominators

Let $0 < \xi < 1$ be an irrational number with regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then the following identity holds for almost all ξ:

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} b_k \right) = \frac{1 + o(1)}{\ln(2)}\, n \ln(n) + \theta_n \max_{1\leq k\leq n} b_k,$$

where θ_n is a $(0,\ 1)$-valued random variable.

SumOfRegularContinuedFractionPartialDenominatorsDivergence

Let $0 < \xi < 1$ be an irrational number with the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then the following identity holds for almost all ξ and any $0 \le \varepsilon < 1$:

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} b_k}{n \ln^{\varepsilon}(n)} = \infty.$$

SumOfRegularContinuedFractionPartialDenominatorsLimSup

Let $0 < \xi < 1$ be an irrational number with the regular continued fraction expansion

$$\xi = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}.$$

Then the following identity holds for almost all ξ:

$$\limsup_{n \to \infty} \frac{1}{d(n)} \sum_{k=1}^{n} b_k = \frac{1}{\ln(2)},$$

where

$$d(n) = \kappa(n) \ln^2(\kappa(n)) \exp\left(\kappa(n) \ln^2(\kappa(n))\right)$$

and

$$\kappa(n) = \exp\left(2\, W\left(\frac{1}{2} \sqrt{\ln(n)}\right)\right).$$

SumOfRestrictedDenominatorContinuedFractions

Let F_k be the set of all infinite regular continued fractions with partial denominators between 1 and k:

$$F_k = \left\{\xi : \xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j} \bigwedge b_j \in \mathbb{Z}^+ \bigwedge 1 \le b_j \le k\right\}.$$

Then the following identities hold for sums of elements of F_k:

$F_3 + F_4 = \mathbb{R} \bmod 1$

$F_2 + F_7 = \mathbb{R} \bmod 1$

$F_2 + F_2 + F_4 = \mathbb{R} \bmod 1$

$F_2 + F_3 + F_3 = \mathbb{R} \bmod 1.$

SzaszContinuedFractionConvergence

Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n}{1}$$

be a generalized continued fraction,

$x_n = |a_n|$

$y_n = |a_n - \mathrm{Re}(a_n)|$

$s = \sum_{n=1}^{\infty} x_n$, and $t = \sum_{n=1}^{\infty} y_n$. Given s converges and $t \le 2$, then ξ converges.

TauberianTheoremForGrommerFractions

If for some $R > 0$ a Grommer fraction ξ converges for all $|z| \ge R$, then the power series $P(z) = \sum_{n=0}^{\infty} c_n z^{-n-1}$ associated with ξ also converges for all $|z| \ge R$.

TauFractionsHaveBothRepresentationAndApproximationProperties

A continued fraction that represents uniquely all real numbers so that the finite continued fraction represents the elements of an algebraic number field, and conversely, every element of the number field is represented by a finite continued fraction is said to have the representation property.

A number field is said to have the approximation property if for every "irrational" α,

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{k\,Q^2}$$

is satisfied by infinitely many rational elements P/Q of the number field and k is a positive fixed constant.

The algebraic number field generated by ϕ has both the reprsentation and approximation properties. The elements of this number field have the form

$$\frac{a + b\,\phi}{c + d\,\phi}$$

for a, b, c, and d integers and c, d not both 0. The associated continued fractions, known as τ-fractions, have the form

$$r_0 + \frac{\varepsilon_1}{r_1\,\phi +}\;\frac{\varepsilon_2}{r_2\,\phi +}\;\ldots$$

where $\varepsilon_1 = \pm 1$, r_0 is any integer, and the other r_i are positive integers. The representation is unique as long as the rule that if $r_1\,\phi + \varepsilon_1 < 1$, then $r_{i+1} \geq 2$ is observed.

TechnicalLemmaForLimitPeriodicContinuedFractions1

Let $\{d_n\}$, $\{r_n\}$ be sequences of positive numbers. Then the inequality $r_{n-1} - r_n \geq 2\,d_n + 2\,r_n\,r_{n-1}$ is satisfied by

$$r_n = \begin{cases} \frac{1-\beta}{2\,(2\,n+1)} & \text{for } d_n = \frac{1-\beta^2}{4\left(4\,n^2-1\right)},\ \ 0 \leq \beta \leq 1,\ \ n \geq 1 \\ \frac{d}{n^\alpha} & \text{for } d_n = \frac{d}{2\,n^{\alpha+1}},\ \alpha > 1,\ d > 0,\ \ (n-1)^\alpha\,(\alpha-1) > 2\,d_n \\ \frac{3\,r^{n+1}}{1-r} & \text{for } d_n = r^n,\ \ 0 < r < 1,\ \ \left(1 - r^2\right) > 18\,r^{n+1}. \end{cases}$$

TechnicalLemmaForLimitPeriodicContinuedFractions2

If the limit periodic continued fraction $\xi = \mathrm{K}(b_n/1) = [0; b_1, b_2, \ldots]$ is such that $\left|b_n - \left(-\frac{1}{4}\right)\right| \leq \frac{1}{4(4n^2-1)}$ for all $n = 1, 2, \ldots$ and if d_n satisfies one of the conditions

$$d_n = \begin{cases} \frac{1-\beta^2}{4(4n^2-1)} & \text{for } 0 \leq \beta \leq 1,\ n \geq 1 \\ \frac{d}{2n^{\alpha+1}} & \text{for } \alpha > 1,\ d > 0,\ (n-1)^\alpha(\alpha-1) > 2\,d_n \\ r^n & \text{for } 0 < r < 1,\ \left(1-r^2\right) > 18\,r^{n+1}, \end{cases}$$

then

$$\left|f_k^{(n)} - g_k\right| \leq r_n,$$

where $f_k^{(n)} = [0; b_{n+1}, b_{n+2}, \ldots, b_{n+k}]$, $g_k = \left[0; \underbrace{-\frac{1}{4}, -\frac{1}{4}, \ldots, -\frac{1}{4}}_{k\text{ terms}}\right]$, and where r_n satisfies

$$r_n = \begin{cases} \frac{1-\beta}{2(2n+1)} & \text{for } d_n = \frac{1-\beta^2}{4(4n^2-1)},\ 0 \leq \beta \leq 1,\ n \geq 1 \\ \frac{d}{n^\alpha} & \text{for } d_n = \frac{d}{2n^{\alpha+1}},\ \alpha > 1,\ d > 0,\ (n-1)^\alpha(\alpha-1) > 2\,d_n \\ \frac{3r^{n+1}}{1-r} & \text{for } 0\ d_n = r^n,\ 0 < r < 1,\ \left(1-r^2\right) > 18\,r^{n+1}. \end{cases}$$

TechnicalLemmaForLimitPeriodicContinuedFractions3

Suppose that $\xi = \mathrm{K}(b_n/1) = [0; b_1, b_2, \ldots]$ is a limit periodic continued fraction for which $\left|b_n - \left(-\frac{1}{4}\right)\right| \leq \frac{1}{4(4n^2-1)}$ and suppose that the values d_n satisfy one of the conditions

$$d_n = \begin{cases} \frac{1-\beta^2}{4(4n^2-1)} & \text{for } 0 \leq \beta \leq 1,\ n \geq 1 \\ \frac{d}{2n^{\alpha+1}} & \text{for } \alpha > 1,\ d > 0,\ (n-1)^\alpha(\alpha-1) > 2\,d_n \\ r^n & \text{for } 0 < r < 1,\ \left(1-r^2\right) > 18\,r^{n+1}, \end{cases}$$

Then

$$\left|1 - \frac{-\frac{1}{2}}{f^{(n)}}\right| \leq \frac{4\,d_{n+1} + 2\,r_{n+1}}{1 - 4\,d_{n+1}},$$

where $f^{(n)} = [0; b_{n+1}, b_{n+2}, \ldots]$ and where

$$r_n = \begin{cases} \frac{1-\beta}{2(2n+1)} & \text{for } d_n = \frac{1-\beta^2}{4(4n^2-1)},\ 0 \leq \beta \leq 1,\ n \geq 1 \\ \frac{d}{n^\alpha} & \text{for } d_n = \frac{d}{2n^{\alpha+1}},\ \alpha > 1,\ d > 0,\ (n-1)^\alpha(\alpha-1) > 2\,d_n \\ \frac{3r^{n+1}}{1-r} & \text{for } 0\ d_n = r^n,\ 0 < r < 1,\ \left(1-r^2\right) > 18\,r^{n+1}. \end{cases}$$

TechnicalLemmaForMeromorphicExtensionOfJFractions1

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z+b_1-\cfrac{a_2}{z+b_2-\cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose without loss of generality that $\lim a_n = 1/4$, $\lim b_n = 0$. For notational convenience, let u_j and v_j be the related terms defined so that $u_j = 2\, b_j$, $j \geq 0$, and $v_j = 1 - 4\, a_j$, $j \geq 1$, with $v_0 = 0$. Also, for natural numbers $j, n, r, k+1 \in \mathbb{Z}^+$ and for complex $\omega \in \mathbb{C}$ with $w = \omega^2$, define the terms $c_{k,j}(\omega)$, $S_k^{(n)}(\omega)$ to be

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\, u_j\left(1 - w^{j-k}\right) + w\, v_j\left(1 - w^{j-k-1}\right)\right) \text{ for } j, k \geq -1$$

and

$$S_k^{(n)}(\omega) = 1 - w^{n-k} + \sum_{r=1}^{n-k-1} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega)\left(1 - w^{n-j_r}\right),$$

respectively, where $c_{k,j}(\pm 1) = \pm (j-k)\, u_j + (j-k-1)\, v_j$ by definition. If $C_n(\omega)$, $D_n(\omega)$ are terms which satisfy the recursions $C_0(\omega) = D_{-1}(\omega) = 0$, $C_1(\omega) = D_0(\omega) = 1 - w$, and

$$C_{n+1}(\omega) - C_n(\omega) = w\,(C_n(\omega) - C_{n-1}(\omega)) + u_n\, \omega\, C_n(\omega) + v_n\, w\, C_{n-1}(\omega), \text{ for } n \geq 1$$

$$D_{n+1}(\omega) - D_n(\omega) = w\,(D_n(\omega) - D_{n-1}(\omega)) + u_n\, \omega\, D_n(\omega) + v_n\, w\, D_{n-1}(\omega), \text{ for } n \geq 0,$$

then for all $n \geq 1$, $C_n(\omega) = S_0^{(n)}(\omega)$ and for all $n \geq 0$, $D_n(\omega) = S_{-1}^{(n)}(\omega)$.

TechnicalLemmaForMeromorphicExtensionOfJFractions2

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z+b_1-\cfrac{a_2}{z+b_2-\cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$, and

$$\sum_{j=1}^{\infty}\left(\left|\frac{a_j - 1}{4}\right| + \left|b_j\right|\right) < \infty.$$

For an arbitrary complex number $\omega \in \mathbb{C}$, let $w = \omega^2$, and for natural numbers $n, r, k+1 \in \mathbb{Z}^+$, define the terms $c_{k,j}(\omega)$, $S_{k,r}^{(n)}(\omega)$ to be

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\, u_j\left(1 - w^{j-k}\right) + w\, v_j\left(1 - w^{j-k-1}\right)\right)$$

and

$$S_{k,r}^{(n)}(\omega) = \sum_{j=k+1}^{n-r} c_{k,j}(\omega)\, S_{j,r-1}^{(n)}(\omega) \text{ for } r \geq 1,\;\; k \geq -1,\;\; n > k + r,$$

respectively, where $S_{k,0}^{(n)} = 1 - w^{n-k}$ for $n > k \geq -1$ and where $c_{k,j}(\pm 1) = \pm (j - k)\, u_j + (j - k - 1)\, v_j$ by definition. Under these hypotheses:

1. If $|\omega| \leq 1$, $\omega \neq \pm 1$, $r \geq 1$, $k \geq -1$, and $n > k + r$, then

$\left|S_{k,r}^{(n)}(\omega)\right| \leq 2\, |1 - w|^{-r}\, \rho_k(1)\, \rho_{k+1}(1) \cdots \rho_{k+r-1}(1)$,

where $\rho_k(R) = \sum_{j=k+1}^{\infty} \left(\left|u_j\right| R^{1/2} \left(1 + R^{j-k}\right) + \left|v_j\right| \left(R + R^{j-k}\right)\right)$ for $u_j = 2\, b_j$, $j \geq 0$, and $v_j = 1 - 4\, a_j$, $j \geq 1$, with $v_0 = 0$.

2. For each $k \geq -1$, $r \geq 1$, the r-fold series $S_{k,r}$ defined by

$$S_{k,r}(\omega) = \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega)$$

converges absolutely and uniformly on compact subsets of $|\omega| \leq 1$, $\omega \neq \pm 1$, and satisfies

$\left|S_{k,r}(\omega)\right| \leq |1 - w|^{-r}\, \rho_k(1)\, \rho_{k+1}(1) \cdots \rho_{k+r-1}(1)$

for $|\omega| \leq 1$, $\omega \neq \pm 1$. Therefore, $S_{k,r}$ is holomorphic for $|\omega| < 1$, is continuous, and satisfies $S_{k,0}(\omega) = 1$ and, for $r \geq 1$, $S_{k,r}(\omega) = \sum_{j=k+1}^{\infty} c_{k,j}(\omega)\, S_{j,r-1}(\omega)$.

3. For each $k \geq -1$, $S_k(\omega)$ converges uniformly and absolutely on compact subsets of $|\omega| \leq 1$, $\omega \neq \pm 1$ where

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega).$$

Therefore, S_k is holomorphic for $|\omega| < 1$, is continuous, and satisfies $S_k(\omega) = \sum_{r=0}^{\infty} S_{k,r}(\omega)$.

4. For each $k \geq -1$, $r \geq 0$, $0 < t < 1$,

$$\lim_{n\to\infty} S_{k,r}^{(n)}(\omega) = S_{k,r}(\omega)$$

and

$$\lim_{n\to\infty} S_k^{(n)}(\omega) = S_k(\omega)$$

where

$$S_k^{(n)}(\omega) = 1 - w^{n-k} + \sum_{r=1}^{n-k-1} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega) \left(1 - w^{n-j_r}\right).$$

TechnicalLemmaForMeromorphicExtensionOfJFractions3

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z + b_1 - \cfrac{a_2}{z + b_2 - \cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$, and

$$\sum_{j=1}^{\infty} j\left(\left|\frac{a_j - 1}{4}\right| + |b_j|\right) < \infty.$$

Under these hypotheses and for $\omega \in \mathbb{C}$ arbitrary, it follows that:

1. For each $k \geq -1$, $r > 0$, the r-fold series $S_{k,r}$ defined by

$$S_{k,r}(\omega) = \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega)$$

converges absolutely and uniformly for $|\omega| \leq 1$ and satisfies $|S_{k,r}(\omega)| \leq \sigma_k\, \sigma_{k+1} \cdots \sigma_{k+r-1}$ for $|\omega| \leq 1$, where

$$\sigma_k = \sum_{j=k+1}^{\infty} \left((j-k)\,|u_j| + (j-k-1)\,|v_j|\right),$$

$u_j = 2\, b_j$ for $j \geq 0$, and $v_j = 1 - 4\, a_j$ for $j \geq 1$. Hence, $S_{k,r}$ is continuous for $|\omega| \leq 1$ and satisfies $S_{k,0}(\omega) = 1$ and, for $r \geq 1$, $S_{k,r}(\omega) = \sum_{j=k+1}^{\infty} c_{k,j}(\omega)\, S_{j,r-1}(\omega)$. Here,

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\, u_j\left(1 - w^{j-k}\right) + w\, v_j\left(1 - w^{j-k-1}\right)\right)$$

with $c_{k,j}(\pm 1) = \pm (j-k)\, u_j + (j-k-1)\, v_j$ by definition.

2. For each $k \geq -1$, S_k converges absolutely and uniformly for all $|\omega| \leq 1$ where

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega).$$

3. For each $k \geq -1$, $S_k^{(n)}$ satisfies $S_k^{(n)}(\pm 1) = 0$ and

$$\lim_{n\to\infty} \left[\lim_{\omega\to\pm 1} S_k^{(n)}(\omega)/(1-\omega)\right] \Big/ n = S_k(\pm 1)$$

where for $w = \omega^2$,

$$S_k^{(n)}(\omega) = 1 - w^{n-k} + \sum_{r=1}^{n-k-1} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega)\left(1 - w^{n-j_r}\right).$$

TechnicalLemmaForMeromorphicExtensionOfJFractions4

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z+b_1 - \cfrac{a_2}{z+b_2 - \cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$, and

$$\sum_{j=1}^{\infty} \left(\left| \frac{a_j - 1}{4} \right| + |b_j| \right) < \infty.$$

Furthermore, let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$. Under these hypotheses, the following results hold:

1. For every $k \geq -1$, $\lim_{n\to\infty} X_k^{(n)}(\omega) = S_k(\omega)$ holds uniformly on compact subsets of $|\omega| \leq 1$, $\omega \neq \pm 1$, where for $n > k \geq -1$

$$X_k^{(n)}(\omega) = 1 + \sum_{r=1}^{n-k-1} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

where $c_{k,j}(\pm 1) = \pm (j-k)\, u_j + (j-k-1)\, v_j$ by definition, and where

$$c_{k,j}(\omega) = (1-w)^{-1} \left(\omega\, u_j \left(1 - w^{j-k}\right) + w\, v_j \left(1 - w^{j-k-1}\right)\right).$$

2. If in addition to the conditions in (1.) above $\sum_{j=1}^{\infty} j\left(\left|a_j - 1/4\right| + |b_j|\right) < \infty$, then $X_k^{(n)}(\omega) \to S_k(\omega)$ uniformly on $|\omega| \leq 1$ as $n \to \infty$.

3. For fixed $k \geq -1$, $S_k^{(n)}(\omega) = S_k(\omega) - w^{n-k}\, S_k(\overline{\omega}) + O(1)$ on $|\omega| = 1$ as $n \to \infty$ whenever the conditions in (2.) are met. Here,

$$S_k^{(n)}(\omega) = 1 - w^{n-k} + \sum_{r=1}^{n-k-1} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega) \left(1 - w^{n-j_r}\right).$$

4. For fixed $k \geq -1$ and for each ω satisfying $|\omega| = 1$, $\omega \neq \pm 1$, $L(\omega) = \lim_{n\to\infty} S_k^{(n)}(\omega)$ exists. Moreover, $L(\omega) = S_k(\omega)$ if and only if $S_k(\overline{\omega}) = 0$.

TechnicalLemmaForMeromorphicExtensionOfJFractions5

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z + b_1 - \cfrac{a_2}{z + b_2 - \cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$, and

$$\sum_{j=1}^{\infty} \left(\left| \frac{a_j - 1}{4} \right| + \left| b_j \right| \right) R^j < \infty$$

for some $R > 1$. Furthermore, let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$ and for notational convenience, let $u_j = 2\, b_j$ for $j \geq 0$ and let $v_j = 1 - 4\, a_j$ for $j \geq 1$. Then the following results hold:

1. For each $k \geq -1$ and $r > 0$, $S_{k,r}$ converges absolutely and uniformly, and for $|\omega| \leq R^{1/2}$ satisfies

$$\left| S_{k,r}(\omega) \right| \leq (R-1)^{-r} \rho_k(R)\, \rho_{k+1}(R) \cdots \rho_{k+r-1}(R).$$

Here,

$$S_{k,r}(\omega) = \sum_{k < j_1 < j_2 < \cdots < j_r < n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1-w)^{-1} \left(\omega\, u_j \left(1 - w^{j-k}\right) + w\, v_j \left(1 - w^{j-k-1}\right) \right)$$

with $c_{k,j}(\pm 1) = \pm (j-k)\, u_j + (j-k-1)\, v_j$ by definition, and

$$\rho_k(R) = \sum_{j=k+1}^{\infty} \left(\left| u_j \right| R^{1/2} \left(1 + R^{j-k}\right) + \left| v_j \right| \left(R + R^{j-k}\right) \right).$$

In particular, for all $k \geq -1$ and $r > 0$, $S_{k,r}$ is holomorphic for $|\omega| < R^{1/2}$, is continuous for $|\omega| \leq R^{1/2}$, and satisfies $S_{k,0}(\omega) = 1$ and, for $r \geq 1$, $S_{k,r}(\omega) = \sum_{j=k+1}^{\infty} c_{k,j}(\omega)\, S_{j,r-1}(\omega)$ for $|\omega| \leq R^{1/2}$.

2. For each $k \geq -1$, the function $S_k(\omega)$ defined by

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k < j_1 < j_2 < \cdots < j_r < n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega)$$

converges absolutely and uniformly for $|\omega| \leq R^{1/2}$. In particular, for all $k \geq -1$, S_k is holomorphic for $|\omega| < R^{1/2}$, is continuous for $|\omega| \leq R^{1/2}$, and satisfies $S_k(\omega) = \sum_{r=0}^{\infty} S_{k,r}(\omega)$ for $|\omega| \leq R^{1/2}$.

TechnicalLemmaForMeromorphicExtensionOfJFractions6

Let $f(z)$ be a J-fraction of the form

$$f(z)=\cfrac{1}{z+b_0-\cfrac{a_1}{z+b_1-\cfrac{a_2}{z+b_2-\cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$. Furthermore, let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$ and for notational convenience, let $u_j = 2\, b_j$ for $j \geq 0$ and let $v_j = 1 - 4\, a_j$ for $j \geq 1$. Let C, D be functions defined such that $C(\omega) = S_0(\omega)$, $D = S_{-1}(\omega)$ for

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1-w)^{-1} \left(\omega\, u_j \left(1-w^{j-k}\right) + w\, v_j \left(1-w^{j-k-1}\right)\right),$$

with $c_{k,j}(\pm 1) = \pm (j-k)\, u_j + (j-k-1)\, v_j$ by definition. Further, define $C_n(\omega)$, $D_n(\omega)$ to be the functions which satisfy the recursions

$C_0(\omega) = D_{-1}(\omega) = 0$, $C_1(\omega) = D_0(\omega) = 1 - w$,

$C_{n+1}(\omega) - C_n(\omega) = w\,(C_n(\omega) - C_{n-1}(\omega)) + u_n\, \omega\, C_n(\omega) + v_n\, w\, C_{n-1}(\omega)$, $n \geq 1$,

$D_{n+1}(\omega) - D_n(\omega) = w\,(D_n(\omega) - D_{n-1}(\omega)) + u_n\, \omega\, D_n(\omega) + v_n\, w\, D_{n-1}(\omega)$, $n \geq 0$.

Under these hypotheses and for every $k \geq 1$, the identity

$$C(\omega)\, D_k(\omega) - D(\omega)\, C_k(\omega) = S_k(\omega)\, w^k\, (1-w) \prod_{j=1}^{k} \left(1 - v_j\right)$$

holds under the following conditions:

1. For $|\omega| \leq R^{1/2}$ if

$$\sum_{j=1}^{\infty} \left(\left| \frac{a_j - 1}{4} \right| + \left| b_j \right| \right) R^j < \infty$$

for some $R > 1$.

2. For $|\omega| \leq 1$, $\omega \neq \pm 1$ if

$$\sum_{j=1}^{\infty} \left(\left| \frac{a_j - 1}{4} \right| + \left| b_j \right| \right) < \infty.$$

3. For $|\omega| \leq 1$ if

$$\sum_{j=1}^{\infty} j \left(\left| a_j - 1/4 \right| + \left| b_j \right| \right) < \infty.$$

In each of the above cases, the functions $C(\omega)$, $D(\omega)$ have no common zeros.

TexanTheorem

Let ξ be the positive number $0<\xi<1$ with regular continued fraction expansion

$$\xi = \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_j}$$

with $a_j \in A$ where A is a finite subset of positive integers. Let $\mathcal{D}(A)$ be the Hausdorff dimension of all ξ for a given set A. Let $\mathcal{D}$ be the set of all possible values of $\mathcal{D}(A)$ for all possible A. Then the Texan theorem (originally a conjecture but subsequently proven) states that $\mathcal{D}$ is a dense subset of $[0,\ 1]$.

TheoremForConvergentSubsequenceForPadeTableRowsOfFunctionsWithFinitePoles

Let f be a meromorphic function, and $D(m)$ be the largest complex disk where f has less than or equal to m poles. Let $T_{m,n}$ be the m th row Padé approximants, R_m be the radius of $D(m)$, a be an element of $\mathbb{C}-0$, V_m be the poles of f in $D(m)$, and K any compact set in $D(m)$ disjoint from V_m. Then $R_m=\infty$ and there is a subsequence p_i such that for any K, T_{m,p_n} converges uniformly on K.

TheoremForConvergentSubsequenceOfBoundedRowsOfPadeTableForEntireFunctions

Let f be an entire function set, and λ be the order of f. Let $T_{m,n}$ be the m th row Padé approximants, and K be any compact set. Then given $(-1+m)\,m\,\lambda<2$, there is a subsequence p_i such that for any K, T_{m,p_n} converges uniformly on K.

TheoremForMeromorphicExtensionOfGeneralAnalyticFractions1

Let $F(z)$ be a general analytic limit periodic continued fraction of the form

$$F(z) = \cfrac{1}{\lambda(z)+b_0(z)-\cfrac{a_1(z)}{\lambda(z)+b_1(z)-\cfrac{a_2(z)}{\lambda(z)+b_2(z)-\cfrac{a_3(z)}{\ddots}}}}$$

where $a_n(z)\not\equiv 0$, $b_{n-1}(z)$ and $\lambda(z)$ are holomorphic functions of z in a region $G\subset\mathbb{C}$ for $n\geq 1$, and where $\lim_{n\to\infty} a_n(z)=1/4$, $\lim_{n\to\infty} b_n(z)=0$ hold uniformly on each compact subset of G. Assume further that the partial quotients of F satisfy

$$\sum_{j=1}^{\infty}\left(\left|a_j(z)-1/4\right|+\left|b_j(z)\right|\right)<\infty$$

uniformly on compact subsets of G, that G^* is defined so that $G^* = G\backslash S$ where $S = \{z \in G : \lambda(z) \in [-1, 1]\}$, and that the region $\emptyset \neq G_0 \subset G^*$ is such that $\lambda\left(G_0 \bigcup \left(\overline{G_0} \bigcap S\right)\right) \subset Y$

where $\overline{G_0}$ denotes the closure of G_0 in $\mathbb{C}$ and where $Y = \mathbb{C}^* \bigcup U$ or $Y = \mathbb{C}^* \bigcup L$ for $\mathbb{C}^* = \mathbb{C}\backslash[-1, 1]$ and for U, respectively L, defined to be the upper, respectively lower, boundary of the cut $[-1, 1]$ of $\mathbb{C}^*$ considered as disjoint subsets of $\mathbb{C}^{**}$ where $\mathbb{C}^{**}$ is defined to be the complete 2-sheeted Riemannian surface obtained by analytic extension of ω from $\mathbb{C}^*$ across $[-1, 1]$ into a second copy of $\mathbb{C}^*$. Under this construction, the following claims hold:

1. Let $\hat{A}(z) = 2\,\hat{\omega}(z)\,C(z, \hat{\omega}(z))$, $\hat{B}(z) = D(z, \hat{\omega}(z))$ be functions defined in terms of $C(z, \omega) = C(\omega) = S_0(\omega)$, $D(z, \omega) = D(\omega) = S_{-1}(\omega)$, where

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\, u_j\left(1-w^{j-k}\right) + w\, v_j\left(1-w^{j-k-1}\right)\right)$$

with $c_{k,j}(\pm 1) = \pm(j-k)\,u_j + (j-k-1)\,v_j$ by definition. Then:

(a) The series for $\hat{A}(z)$ and $\hat{B}(z)$ converge uniformly and absolutely on compact subsets of $G_0 \bigcup \left(\overline{G_0} \bigcap S\right)$.

(b) The functions $\hat{A}(z)$, $\hat{B}(z)$ are holomorphic on G_0 and can be extended continuously onto $G_0 \bigcup \left(\overline{G_0} \bigcap S\right)$ where the extensions have no common zeros there.

(c) If $B \not\equiv 0$ on G_0, then F converges uniformly on compact subsets of $G_0 \big\backslash \left\{z \in G_0 : \hat{B}(z) = 0\right\}$ to $\hat{A}(z)/\hat{B}(z)$.

(d) If $\hat{B}(z) \equiv 0$ on G_0, then $\hat{A}(z) \neq 0$ on G_0 and so $F(z)$ diverges to ∞ on G_0.

2. For each fixed $z \in S = \lambda^{-1}[-1, 1]$ with $\lambda(z) \neq \pm 1$, the continued fraction representation of $F(z)$ diverges. More precisely, if $\hat{\omega} = e^{i\,\vartheta(z)}$ for $\vartheta(z) \neq k\,\pi$ a real number, $k \in \mathbb{Z}$, and if

$$M_z(\zeta) = 2\left(\hat{\omega}\, C(z, \hat{\omega}) - \zeta\, \hat{\omega}^{-1}\, C\left(z, \hat{\omega}^{-1}\right)\right)\left(D(z, \hat{\omega}) - \zeta\, D\left(z, \hat{\omega}^{-1}\right)\right)^{-1}$$

denotes a M bius transform in ζ, then the nth approximant of F at z equals $M_z\left(e^{i\,2\,(n+1)\,\vartheta(z)}\right) + O(1)$ as $n \to \infty$. Thus, for fixed z, the nth approximant of $A(z)/B(z)$ of F lie on the image of the unit circle under $M_z(\zeta)$ which is a straight line if and only if $\left|D\left(z, e^{i\,\vartheta(z)}\right)\right| = \left|D\left(z, e^{-i\,\vartheta(z)}\right)\right|$.

3. If additionally $\sum_{j=1}^{\infty} j\left(\left|a_j(z)-1/4\right| + \left|b_j(z)\right|\right) < \infty$ uniformly on each compact subset of G and if G_0 is a subset of Z where $Z = \mathbb{C} \bigcup U \bigcup \{-1, 1\}$ or $Z = \mathbb{C} \bigcup L \bigcup \{-1, 1\}$, then the result of (a.) above holds for G_0. Moreover, for each $z \in S$ with $\lambda(z) = \pm 1$, $F(z) = \pm 2\,C(z, \pm 1)/D(z, \pm 1)$ where $C(z, \pm 1)$, $D(\pm 1, z)$ do not vanish simultaneously.

TheoremForMeromorphicExtensionOfGeneralAnalyticFractions2

Let $F(z)$ be a general analytic limit periodic continued fraction of the form

$$F(z) = \cfrac{1}{\lambda(z) + b_0(z) - \cfrac{a_1(z)}{\lambda(z) + b_1(z) - \cfrac{a_2(z)}{\lambda(z) + b_2(z) - \cfrac{a_3(z)}{\ddots}}}}$$

where $a_n(z) \not\equiv 0$, $b_{n-1}(z)$ and $\lambda(z)$ are holomorphic functions of z in a region $G \subset \mathbb{C}$ for $n \geq 1$, and where $\lim_{n\to\infty} a_n(z) = 1/4$, $\lim_{n\to\infty} b_n(z) = 0$ hold uniformly on each compact subset of G. Assume further that the partial quotients of F satisfy

$$\sum_{j=1}^{\infty} \left(\left| a_j(z) - 1/4 \right| + \left| b_j(z) \right| \right) R^j < \infty$$

uniformly on compact subsets of G for some $R > 1$, that $G^* = G \setminus S$ where $S = \{z \in G : \lambda(z) \in [-1, 1]\}$, and that G_0^* is a fixed component of G^*. Next, define $\hat{\omega}(z)$ on each component of G^* so that $\hat{\omega}(z) = \omega(\lambda(z))$ where $\omega(z) = z - \left(z^2 - 1\right)^{1/2}$ with roots chosen positive for $z > 1$, $z \in \mathbb{C} \setminus [-1, 1]$, and let G^{**} be defined to be the 2-sheeted Riemannian surface of $\hat{\omega}(z)$ over G obtained by analytic extension of $\hat{\omega}$ from each component of G^* across S into a second copy of G with G_0^{**} the smallest subregion of G^{**} with $G_0^* \subset G_0^{**}$ such that no point in G_0^{**} lies above $S(R)$ but that the boundary $\partial_R G_0^{**} = \partial G_0^{**} \cap G^{**}$ of G_0^* lies above $S(R)$. Here, for $R > 1$, $S(R) = \lambda^{-1}(E(R)) \subset G$ where $E(R)$ denotes the ellipse

$$E(R) = \left\{ z \in \mathbb{C} : \left(\mathrm{Re}(z) / \left(R^{1/2} + R^{-1/2}\right) \right)^2 + \left(\mathrm{Im}(z) / \left(R^{1/2} - R^{-1/2}\right) \right)^2 = \frac{1}{4} \right\}.$$

From this, the following hold:

1. Let $\hat{A}(z) = 2\,\hat{\omega}(z)\, C(z, \hat{\omega}(z))$, $\hat{B}(z) = D(z, \hat{\omega}(z))$ be functions defined in terms of $C(z, \omega) = C(\omega) = S_0(\omega)$, $D(z, \omega) = D(\omega) = S_{-1}(\omega)$, where

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k < j_1 < j_2 < \cdots < j_r < n} c_{k, j_1}(\omega)\, c_{j_1, j_2}(\omega) \cdots c_{j_{r-1}, j_r}(\omega),$$

$$c_{k,j}(\omega) = (1 - w)^{-1} \left(\omega\, u_j \left(1 - w^{j-k}\right) + w\, v_j \left(1 - w^{j-k-1}\right) \right)$$

with $c_{k,j}(\pm 1) = \pm (j - k)\, u_j + (j - k - 1)\, v_j$ by definition. Then:

(a) The explicit series representations for $\hat{A}$, $\hat{B}$ converge absolutely and uniformly on compact subsets of $G_0^{**} \cup \partial_R G_0^{**}$.

(b) $\hat{A}$ and $\hat{B}$ can be extended analytically from G_0^* across S into $G_0^* *$.

(c) $\hat{A}$ and $\hat{B}$ and can be extended continuously onto $G_0^{**} \cup \partial_R G_0^{**}$ and the extensions have no zeros there.

2. The branch points of $\ddot{\omega}(z)$ are the algebraic first order branch points for the extended meromorphic function $F(z) = \hat{A}(z)/\hat{B}(z)$ proved $\hat{B} \not\equiv 0$ on G_0^*.

3. At each $z_0 \in S$ with $\lambda(z_0) = \pm 1$ of even order, $\hat{A}$ and $\hat{B}$ consist of two separate holomorphic branches in a neighborhood around z_0.

4. If additionally

$$\sum_{j=1}^{\infty} \left(\left| a_j(z) - 1/4 \right| + \left| b_j(z) \right| \right) R^j < \infty$$

uniformly on compact subsets of G for all $R > 1$, then for each component of G^*, $\hat{A}$ and $\hat{B}$ can be extended analytically across S into the whole Riemannian surface G^{**}. Moreover, if $\hat{B} \not\equiv 0$, then the extended $F = \hat{A}/\hat{B}$ is meromorphic on G^{**}.

TheoremForMeromorphicExtensionOfGeneralAnalyticFractions3

Let $F(z)$ be a general analytic limit periodic continued fraction of the form

$$F(z) = \cfrac{1}{\lambda(z) + b_0(z) - \cfrac{a_1(z)}{\lambda(z)+b_1(z) - \cfrac{a_2(z)}{\lambda(z)+b_2(z) - \cfrac{a_3(z)}{\ddots}}}}$$

where $a_n(z) \not\equiv 0$, $b_{n-1}(z)$ and $\lambda(z)$ are holomorphic functions of z in a region $G \subset \mathbb{C}$ for $n \geq 1$, and where $\lim_{n\to\infty} a_n(z) = 1/4$, $\lim_{n\to\infty} b_n(z) = 0$ hold uniformly on each compact subset of G. Assume further that the partial quotients of F satisfy

$$\sum_{j=1}^{\infty} \left(\left| a_j(z) - 1/4 \right| + \left| b_j(z) \right| \right) < \infty$$

uniformly on compact subsets of G, define $G^* = G \backslash S$ where $S = \{z \in G : \lambda(z) \in [-1, 1]\}$, and define the transformation $\hat{\omega}(z)$ on each component of G^* so that $\hat{\omega}(z) = \omega(\lambda(z))$ where $\omega(z) = z - \left(z^2 - 1\right)^{1/2}$ with roots chosen positive for $z > 1$, $z \in \mathbb{C} \backslash [-1, 1]$, and let G^{**} be defined to be the 2-sheeted Riemannian surface of $\hat{\omega}(z)$ over G obtained by analytic extension of $\hat{\omega}$ from each component of G^* across S into a second copy of G. Finally, let G_1^* be a fixed component of G^*, G_1^{**} a subregion of G^{**}, and $H_1^{**} \subset G^{**}$ so that $G_1^* \subsetneq G_1^* * \subset H_1^{**} \subset G^{**}$, and suppose that

$$\sum_{j=1}^{\infty} \left(\left| a_j(z) - 1/4 \right| + \left| b_j(z) \right| \right) |\hat{\omega}(z)| < \infty$$

uniformly on compact subsets of H_1^{**} where $\hat{\omega}$ is assumed to have been extended analytically onto G^{**} with $|\hat{\omega}(z)| < 1$ for $z \in G^*$ and with $\hat{\omega}(z) \neq \pm 1$ for $z \in H_1^{**}$. Under these hypotheses, the following results hold:

1. The explicit series representations for $\hat{A}(z)$ and $\hat{B}(z)$ converge absolutely and uniformly on compact subsets of H_1^{**}.

2. $\hat{A}$ and $\hat{B}$ can be extended analytically from G_1^* across S into G_1^{**}.

3. $\hat{A}$ and $\hat{B}$ can be extended continuously onto H_1^* and the extensions have no common zeros there.

For the above, $\hat{A}(z) = 2\,\hat{\omega}(z)\, C(z, \hat{\omega}(z))$, $\hat{B}(z) = D(z, \hat{\omega}(z))$ are functions defined in terms of $C(z, \omega) = C(\omega) = S_0(\omega)$, $D(z, \omega) = D(\omega) = S_{-1}(\omega)$, where

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1-w)^{-1} \left(\omega\, u_j \left(1 - w^{j-k}\right) + w\, v_j \left(1 - w^{j-k-1}\right)\right)$$

with $c_{k,j}(\pm 1) = \pm (j-k)\, u_j + (j-k-1)\, v_j$ by definition.

TheoremForMeromorphicExtensionOfJFractions1

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z+b_1 - \cfrac{a_2}{z+b_2-\cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$, and

$$\sum_{j=1}^{\infty}\left(\left|\frac{a_j - 1}{4}\right| + \left|b_j\right|\right) < \infty.$$

Furthermore, let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$ and for notational convenience, let $u_j = 2\, b_j$ for $j \geq 0$ and let $v_j = 1 - 4\, a_j$ for $j \geq 1$. Finally, define the functions $C(\omega)$, $D(\omega)$ to be $C(\omega) = S_0(\omega)$, $D = S_{-1}(\omega)$ for

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\, u_j\left(1 - w^{j-k}\right) + w\, v_j\left(1 - w^{j-k-1}\right)\right),$$

with $c_{k,j}(\pm 1) = \pm(j-k)\, u_j + (j-k-1)\, v_j$ by definition. With these assumptions, the following claims hold:

1. Define the functions A^+, A^-, B^+, B^- as follows: $A^+(x) = 2\, e^{-i\vartheta}\, C\left(e^{-i\vartheta}\right)$, $A^-(x) = 2\, e^{i\vartheta}\, C\left(e^{i\vartheta}\right)$, $B^+(x) = D\left(e^{-i\vartheta}\right)$, and $B^-(x) = D\left(e^{i\vartheta}\right)$. Then A^+, A^-, B^+, B^- are continuous on $(-1, 1)$ and for every $x = \cos(\vartheta)$, $\vartheta \in (0, \pi)$, they satisfy

$$A^-(x)\, B^+(x) - A^+(x)\, B^-(x) = 4\, i\left(1 - x^2\right)^{1/2} \prod_{j=1}^{\infty}\left(1 - v_j\right) = 4\, i \sin\vartheta \prod_{j=1}^{\infty}\left(1 - v_j\right) \neq 0.$$

If additionally all a_n, b_n in $f(z)$ are real numbers, then $A^-(x) = \overline{A^+(x)} \neq 0$ and $B^-(x) = \overline{B^+(x)} \neq 0$ for all $x \in (-1, 1)$.

2. For $\lambda \in \mathbb{C}^* = \mathbb{C} \backslash [-1, 1]$, let $\omega(\lambda)$ denote the transformation

$$\omega(\lambda) = \frac{1}{2}\left((\lambda+1)^{1/2} - (\lambda-1)^{1/2}\right)^2$$

with roots assumed to be positive for $\lambda > 1$ and define functions A, B so that

$A(\lambda) = 2\, \omega(\lambda)\, C(\omega(\lambda))$,

$B(\lambda) = D(\omega(\lambda))$.

Defined in this way:

(a) The functions $A, B \not\equiv 0$ are holomorphic on $\mathbb{C}^* \bigcup \{\infty\}$ and can thus be extended continuously onto $\mathbb{C}^* \bigcup U \bigcup L$ where U, respectively L, denotes the upper, respectively lower, boundary of the cut $[-1, 1]$ of $\mathbb{C}^*$ considered as disjoint subsets of $\mathbb{C}^{**}$ where $\mathbb{C}^{**}$ is defined to be the complete 2-sheeted Riemannian surface obtained by analytic extension of ω from $\mathbb{C}^*$ across $[-1, 1]$ into a second copy of $\mathbb{C}^*$. In particular then, $A(\lambda)$ and $B(\lambda)$ approach continuous boundary values of $A^+(\lambda)$, $B^+(\lambda)$, respectively $A^-(\lambda)$, $B^-(\lambda)$ if $\lambda \in \mathbb{C}^*$ approaches $x \in U$, respectively $x \in L$.

(b) A and B do not vanish simultaneously on $\mathbb{C}^* \bigcup U \bigcup L$.

(c) The function $f(\lambda)$ defined to be

$$f(\lambda) = \lim_{n\to\infty} A_n(\lambda)/B_n(\lambda)$$

for A_n/B_n the nth approximant of $f(z)$ satisfies $f(\lambda) = A(\lambda)/B(\lambda)$ uniformly on compact subsets of $\mathbb{C}^* \backslash \{\lambda \in \mathbb{C}^* : B(\lambda) = 0\}$.

3. For $x = \cos\vartheta$, $\vartheta \in (0, \pi)$, the continued fraction $f(x)$ diverges. More precisely, $A_n(x)/B_n(x) = M\left(e^{-i\,2\,(n+1)\,\vartheta}\right) + O(1)$ holds uniformly on compact subsets of $(-1, 1)$ as $n \to \infty$ where

$$M(\zeta) = (A^+(\zeta) - \zeta\,A^-(\zeta))/(B^+(\zeta) - \zeta\,B^-(\zeta))$$

is a Möbius transformation. Thus, for fixed $x \in (-1, 1)$, all $A_n(x)/B_n(x)$ lie asymptotically on the image of the unit circle under $M(\zeta)$ which is a straight line if and only if $|B^+(x)| = |B^-(x)|$.

4. If $\sum_{j=1}^{\infty} j\left(\left|a_j - 1/4\right| + \left|b_j\right|\right) < \infty$ holds, then so does (1.) above. Moreover, A and B can be extended continuously from $\mathbb{C}^* \bigcup U \bigcup L$ into ± 1 and $A(\lambda), B(\lambda) \to A(\pm 1), B(\pm 1)$ as $\lambda \in \mathbb{C}^* \bigcup U \bigcup L \to \pm 1$ where by definition $A(\pm 1) = \pm 2\,C(\pm 1)$, $B(\pm 1) = D(\pm 1)$. Moreover, neither $A(1), B(1)$ nor $A(-1), B(-1)$ vanish simultaneously and

$$\lim_{n\to\infty} A_n(\pm 1)/B_n(\pm 1) = A(\pm 1)/B(\pm 1).$$

TheoremForMeromorphicExtensionOfJFractions2

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z+b_1 - \cfrac{a_2}{z+b_2 - \cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, and suppose that $\lim a_n = 1/4$, $\lim b_n = 0$, and

$$\sum_{j=1}^{\infty}\left(\left|\frac{a_j - 1}{4}\right| + \left|b_j\right|\right) R^j < \infty$$

for some $R > 1$. Furthermore, let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$ and for notational convenience, let $u_j = 2\,b_j$ for $j \geq 0$ and let $v_j = 1 - 4\,a_j$ for $j \geq 1$. Moreover, suppose the functions $C(\omega)$, $D(\omega)$ are defined to be $C(\omega) = S_0(\omega)$, $D = S_{-1}(\omega)$ for

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\,c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\,u_j\left(1 - w^{j-k}\right) + w\,v_j\left(1 - w^{j-k-1}\right)\right),$$

with $c_{k,j}(\pm 1) = \pm(j-k)\,u_j + (j-k-1)\,v_j$ by definition, and suppose

$\lambda \in \mathbb{C}^* = \mathbb{C} \backslash [-1, 1]$ $\omega(\lambda)$ denotes the transformation

$\lambda \in \mathbb{C}^* = \mathbb{C} \setminus [-1, 1]$ $\omega(\lambda)$ denotes the transformation

$$\omega(\lambda) = \frac{1}{2}\left((\lambda+1)^{1/2} - (\lambda-1)^{1/2}\right)^2$$

with roots assumed to be positive for $\lambda > 1$. Under this construction, define the functions A, B so that

$$A(\lambda) = 2\,\omega(\lambda)\,C(\omega(\lambda)),$$

$$B(\lambda) = D(\omega(\lambda)).$$

Then:

1. If U, respectively L, denotes the upper, respectively lower, boundary of the cut $[-1, 1]$ of $\mathbb{C}^*$ considered as disjoint subsets of $\mathbb{C}^{**}$, if $\mathbb{C}^{**}$ is defined to be the complete 2-sheeted Riemannian surface obtained by analytic extension of ω from $\mathbb{C}^*$ across $[-1, 1]$ into a second copy of $\mathbb{C}^*$, and if $E(R) = \left\{|\omega(z)| = R^{1/2}\right\}$ is the ellipse with explicit form

$$E(R) = \left\{z \in \mathbb{C} : \left(\mathrm{Re}(z) \middle/ \left(R^{1/2} + R^{-1/2}\right)\right)^2 + \left(\mathrm{Im}(z) \middle/ \left(R^{1/2} - R^{-1/2}\right)\right)^2 = \frac{1}{4}\right\}.$$

then:

(a) The functions A and B can be extended analytically from $\mathbb{C}^*$ across U and L onto a subregion $|\omega(\lambda)| < R^{1/2}$ of the region $\mathbb{C}^{**}$ whose boundary $|\omega(\lambda)| = R^{1/2}$ on $\mathbb{C}^{**}$ lies above the ellipse $E(R)$.

(b) Onto the boundary $|\omega(\lambda)| = R^{1/2}$, A and B can be extended continuously.

(c) The foci $z = \pm 1$ of the ellipse $E(R)$ are first order algebraic branched points for $f(\lambda) = A(\lambda)/B(\lambda)$.

(d) A and B have no common zeros in the extension $|\omega(\lambda)| \le R^{1/2}$.

2. If $\tilde{A}$, $\tilde{B}$ denote the functions resulting from extending A, B from $\lambda \in \mathbb{C}^*$ across U or L into the point in $\mathbb{C}^{**}$ lying above λ. Then

$$\tilde{A}(\lambda)\,B(\lambda) - A(\lambda)\,\tilde{B}(\lambda) = 4\left(\lambda^2 - 1\right)^{1/2} \prod_{j=1}^{\infty}\left(1 - v_j\right)$$

for all $\lambda \in \mathbb{C}^*$ satisfying $R^{-1/2} \le |\omega(\lambda)| \le R^{1/2}$, where the roots of $\left(\lambda^2 - 1\right)^{1/2}$ are assumed positive for $\lambda > 1$.

3. If

$$\sum_{j=1}^{\infty}\left(\left|\frac{a_j - 1}{4}\right| + \left|b_j\right|\right) R^j < \infty$$

holds for all $R > 1$, then A and B can be extended analytically to functions $\tilde{A}$, $\tilde{B}$ defined on the complete surface $\mathbb{C}^{**}$ in such a way that these extensions satisfy

$$\tilde{A}(\lambda)\,B(\lambda) - A(\lambda)\,\tilde{B}(\lambda) = 4\left(\lambda^2 - 1\right)^{1/2} \prod_{j=1}^{\infty}\left(1 - v_j\right)$$

for all $\lambda \in \mathbb{C}$. In this case, $f(z)$ is meromorphic on $\mathbb{C}^{**}$.

TheoremForMeromorphicExtensionOfJFractions3

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z + b_1 - \cfrac{a_2}{z + b_2 - \cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, where $\lim a_n = 1/4$ and $\lim b_n = 0$ hold, and where $A_n(z)/B_n(z)$ denotes the nth approximant of f. Suppose, too, that a_n, b_n satisfy

$$\sum_{j=1}^{\infty} j\left(\left|a_j - 1/4\right| + \left|b_j\right|\right) < \infty$$

and that $D(\omega) \neq 0$ for all $|\omega| \leq 1$ where $D = S_{-1}(\omega)$ for

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k<j_1<j_2<\cdots<j_r<n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1-w)^{-1}\left(\omega\, u_j \left(1 - w^{j-k}\right) + w\, v_j \left(1 - w^{j-k-1}\right)\right),$$

with $c_{k,j}(\pm 1) = \pm (j-k)\, u_j + (j-k-1)\, v_j$ by definition. If $\phi(x)$ denotes the function

$$\phi(x) = \frac{2}{\pi}\left(1 - x^2\right)^{1/2} \prod_{j=1}^{\infty}\left(1 - v_j\right) \Big/ B^{+}(x)\, B^{-}(x)$$

for $x \in [-1, 1]$ with all roots nonnegative, then:

1. $\phi(x)$ is continuous for all $x \in [-1, 1]$.

2. $\phi(x) \neq 0$ for all $x \in (-1, 1)$ and $\phi(\pm 1) = 0$.

3. For all $\lambda \in \mathbb{C}$,

$$f(\lambda) = \int_{-1}^{1} \phi(x)\,(\lambda - x)^{-1}\, dx.$$

4. If γ is a large circle centered at $z = 0$, then

$$\int_{-1}^{1} B_m(x)\, B_n(x)\, \phi(x)\, dx = \frac{1}{2\pi i} \int_{\gamma} B_m(\lambda)\, B_n(\lambda)\, f(\lambda)\, d\lambda = a_0\, a_1 \cdots a_m\, \delta_{m,n}$$

for m, $n \geq 0$ where $\delta_{i,j}$ denotes Kronecker's delta.

TheoremForMeromorphicExtensionOfJFractions4

Let $f(z)$ be a J-fraction of the form

$$f(z) = \cfrac{1}{z + b_0 - \cfrac{a_1}{z + b_1 - \cfrac{a_2}{z + b_2 - \cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, where $\lim a_n = 1/4$ and $\lim b_n = 0$ hold. Suppose, too, that

$$\sum_{j=1}^{\infty} \left(\left| \frac{a_j - 1}{4} \right| + \left| b_j \right| \right) < \infty,$$

and let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$ where, for convenience, the notation $u_j = 2\, b_j$ for $j \geq 0$ and let $v_j = 1 - 4\, a_j$ for $j \geq 1$ is adopted.

Then:

1. For $x \in (-1, 1)$, $f(\lambda)$ can be written as

$$f(\lambda) = \int_{-a}^{a} (\lambda - x)^{-1}\, d\psi(x)$$

for $\lambda \in \mathbb{C} \backslash [-a, a]$, where ψ is a real-valued nondecreasing function on $[-a, a]$ normalized so that $\psi(x) = \psi(x + 0)$ for all $x \in (-a, a)$.

2. $\psi(x)$ is differentiable and satisfies $\psi'(x) = \phi(x)$ where

$$\phi(x) = \frac{2}{\pi} \left(1 - x^2\right)^{1/2} \prod_{j=1}^{\infty} \left(1 - v_j\right) \Big/ B^+(x)\, B^-(x)$$

for $x \in [-1, 1]$ with all roots nonnegative. Here, B^+, B^- are functions defined by the first substituting $x = \cos\vartheta$, $\vartheta \in (0, \pi)$, and then defining $B^+(x) = D\left(e^{-i\vartheta}\right)$ and $B^-(x) = D\left(e^{i\vartheta}\right)$ where $D = S_{-1}(\omega)$ for

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k < j_1 < j_2 < \cdots < j_r < n} c_{k,j_1}(\omega)\, c_{j_1,j_2}(\omega) \cdots c_{j_{r-1},j_r}(\omega),$$

$$c_{k,j}(\omega) = (1 - w)^{-1} \left(\omega\, u_j \left(1 - w^{j-k}\right) + w\, v_j \left(1 - w^{j-k-1}\right)\right),$$

with $c_{k,j}(\pm 1) = \pm (j - k)\, u_j + (j - k - 1)\, v_j$ by definition.

3. $\psi'(x) = \phi(x)$ is continuous for all $x \in (-1, 1)$.

4. If additionally

$$\sum_{j=1}^{\infty} j\left(\left|a_j - 1/4\right| + \left|b_j\right|\right) < \infty,$$

then $\left(1 - x^2\right)^{1/2} \phi(x)$ is bounded for all $-1 < x < 1$; equivalently, if $x = \cos(\vartheta)$ for $\vartheta \in (0, \pi)$, then $\phi(\cos(\vartheta)) \sin(\vartheta)$ is bounded for $0 < \vartheta < \pi$.

TheoremForMeromorphicExtensionOfJFractions5

Let $f(z)$ be a J-fraction of the form

1

$$f(z) = \cfrac{}{z + b_0 - \cfrac{a_1}{z+b_1 - \cfrac{a_2}{z+b_2 - \cfrac{a_3}{\ddots}}}}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ for $n = 0, 1, 2, \ldots$, where $\lim a_n = 1/4$ and $\lim b_n = 0$ hold, and where $A_n(z)/B_n(z)$ denotes the nth approximant of f. Suppose, too, that

$$\sum_{j=1}^{\infty} \left(\left| \frac{a_j - 1}{4} \right| + \left| b_j \right| \right) < \infty,$$

let $\omega \in \mathbb{C}$ be an arbitrary complex number with $w = \omega^2$ where, for convenience, the notation $u_j = 2\, b_j$ for $j \geq 0$ and let $v_j = 1 - 4\, a_j$ for $j \geq 1$ is adopted, and define $Q_n(\lambda) = B_n(\lambda) \big/ (a_0\, a_1 \cdots a_n)^{1/2}$ where, for each $n \geq 1$,

$$(a_0\, a_1 \cdots a_n)^{1/2} = 2^{-n} \left[\prod_{j=1}^{n} (1 - v_j) \right]^{1/2}$$

is chosen in $\{z \in \mathbb{C} : \arg(z) \in [-\pi/2, \pi/2]\}$. Given this, the following results hold:

1. Define for $\lambda \in \mathbb{C} \backslash [-1, 1]$ the transformation $\omega(\lambda)$ to be

$$\omega(\lambda) = \frac{1}{2} \left((\lambda + 1)^{1/2} - (\lambda - 1)^{1/2} \right)^2$$

with roots assumed positive for $\lambda > 1$. Moreover, let

$$E(R) = \left\{ z \in \mathbb{C} : \left(\mathrm{Re}(z) \big/ \left(R^{1/2} + R^{-1/2} \right) \right)^2 + \left(\mathrm{Im}(z) \big/ \left(R^{1/2} - R^{-1/2} \right) \right)^2 = \frac{1}{4} \right\}.$$

and define the function $B(\lambda) = D(\omega(\lambda))$ for $D = S_{-1}(\omega)$,

$$S_k(\omega) = 1 + \sum_{r=1}^{\infty} \sum_{k < j_1 < j_2 < \cdots < j_r < n} c_{k, j_1}(\omega)\, c_{j_1, j_2}(\omega) \cdots c_{j_{r-1}, j_r}(\omega),$$

$$c_{k,j}(\omega) = (1 - w)^{-1} \left(\omega\, u_j \left(1 - w^{j-k} \right) + w\, v_j \left(1 - w^{j-k-1} \right) \right)$$

with $c_{k,j}(\pm 1) = \pm (j - k)\, u_j + (j - k - 1)\, v_j$ by definition. Under this construction, for fixed $t \in (0, 1)$,

$$(\omega(\lambda))^{n+1}\, Q_n(\lambda) = 2^{-1}\, B(\lambda) \left(\lambda^2 - 1 \right)^{1/2} \left[\prod_{j=1}^{\infty} (1 - v_j) \right]^{1/2} + O(1)$$

as $n \to \infty$. This result holds uniformly for all $|\omega(\lambda)| \leq t$, i.e., uniformly outside the ellipse $E\left(t^{-2}\right)$, where $\left(\lambda^2 - 1 \right)^{1/2}$ is assumed positive for $\lambda > 1$.

2. If $x = \cos(\vartheta)$ for $\vartheta \in (0, \pi)$ and if $n \to \infty$, then

$$2\, i\, Q_n(\cos(\vartheta) \sin(\vartheta)) = \left(D\left(e^{-i\vartheta} \right) e^{i(n+1)\vartheta} - D\left(e^{i\vartheta} \right) e^{-i(n+1)\vartheta} \right) \Big/ \left[\prod_{j=1}^{n} (1 - v_j) \right]^{1/2} + O(1)$$

on compact subsets of $0 < \vartheta < \pi$. If additionally

$$\sum_{j=1}^{\infty} j \left(\left| a_j - 1/4 \right| + \left| b_j \right| \right) < \infty,$$

then the result holds uniformly on $0 \leq \vartheta \leq \pi$.

3. If $x = \cos(\vartheta)$ for $\vartheta \in (0, \pi)$ and if $n \to \infty$, then

$$Q_n^2(\cos(\vartheta)) - Q_{n-1}(\cos(\vartheta))\, Q_{n+1}(\cos(\vartheta)) = D\left(e^{-i\,\vartheta}\right) D\left(e^{i\,\vartheta}\right) \Big/ \prod_{j=1}^{n} \left(1 - v_j\right) + O(1)$$

holds uniformly on all compact subsets of $0 \leq \vartheta \leq \pi$. If additionally

$$\sum_{j=1}^{\infty} j\left(\left|a_j - 1/4\right| + \left|b_j\right|\right) < \infty,$$

then the result holds uniformly on $0 \leq \vartheta \leq \pi$.

4. If in $f(z)$ $b_n \in \mathbb{R}$, $a_n > 0$ for all $n = 0, 1, 2, \ldots$ and if $\Delta(\vartheta) = \arg\left(D\left(e^{i\,\vartheta}\right)\right)$ is chosen as a continuous function of ϑ on $0 < \vartheta < \pi$, then

$$Q_n(\cos(\vartheta)\sin(\vartheta)) = \left|D\left(e^{i\,\vartheta}\right)\right| \sin((n+1)\,\vartheta - \Delta\,(\vartheta)) \Big/ \left[\prod_{j=1}^{n} \left(1 - v_j\right)\right]^{1/2} + O\,(1)$$

uniformly on compact subsets of $0 < \vartheta < \pi$ as $n \to \infty$.

TheoremForMeromorphicExtensionOfTFractions1

Let $T(z)$ be a general limit periodic T-fraction of the form

$$T(z) = \cfrac{1}{1 + d_0\, z - \cfrac{c_1\, z}{1 + d_1\, z - \cfrac{c_2\, z}{1 + d_2\, z - \cfrac{c_3\, z}{\ddots}}}}$$

where c_n, $d_{n-1} \in \mathbb{C}$ are complex numbers with $c_n \neq 0$ for $n \geq 1$ and where $\lim_{n\to\infty} c_n = c \in \mathbb{C}$, $\lim_{n\to\infty} d_n = d \in \mathbb{C}$. Let S denote the divergence line of T. The following cases exhaust all possible values of S.

1. If $d = 0$, then it can be assumed without loss of generality that $c = 1/4$ and hence $S = [1, \infty \subset \mathbb{R}^+$ where $\mathbb{R}^+$ denotes the set of all positive real numbers.

2. Let $c \in \mathbb{C}$ and $d \neq 0$. In this case, it can be assumed without loss of generality that $d = 1$. In each such case, $-1 \in S$ and

$$S = \left\{ \left(t\, c^{1/2} + \left(t^2\, c - 1\right)^{1/2}\right)^2 \,:\, -1 \leq t \leq 1 \right\}$$

holds where all roots are assumed to be positive. The cases are, more precisely:

(a) If $d = 1$ and $c < 0$, then

$$S = \left[-\left((|c| + 1)^{1/2} + |c|^{1/2}\right)^2,\, -\left((|c| + 1)^{1/2} - |c|^{1/2}\right)^2\right] \subset \mathbb{R}^-$$

where $\mathbb{R}^-$ denotes the set of negative real numbers.

(b) If $d = 1$ and $0 < c < 1$, then S is equal to the subarc of the unit circle containing -1 having endpoints $\left(c^{1/2} \pm i\,(1 - c)^{1/2}\right)^2$ in $\mathbb{R}^2 \cong \mathbb{C}$.

(c) If $d = 1$ and $c = 1$, then $S = \{z \in \mathbb{C} \,:\, |z| = 1\}$.

(d) If $d = 1$ and $c > 1$, then $S = I \bigcup \{z \in \mathbb{C} \,:\, |z| = 1\}$ where $I = \left[\left(c^{1/2} - (c - 1)^{1/2}\right)^2,\, \left(c^{1/2} + (c - 1)^{1/2}\right)^2\right] \subset \mathbb{R}^+$ with $1 \in I$.

(e) If $d = 1$ and $c = |c|\, e^{i\,\vartheta}$ with $0 < \vartheta < \pi$, then $S \subset S^\vartheta$ where S^ϑ is the trigonometric spiral

$$S^\vartheta = \left\{ z = r\, e^{i\,\psi} \,:\, r = r(\psi) = \sin((+\vartheta)/2)/\sin((\psi - \vartheta)/2),\ \ \vartheta < \psi < 2\,\pi - \vartheta \right\}$$

with $r(\psi)$ strictly decreasing from ∞ to 0. In particular, S is the subarc of S^ϑ which passes through -1 and has endpoints $r(\psi_0)\, e^{i\,\psi_0}$ and $r(2\,\pi - \psi_0)\, e^{-i\,\psi_0}$ where ψ_0 is characterized by the identity $\cos\psi_0 = |c| - |c - 1|$, $\vartheta < \psi_0 < \pi$.

TheoremForMeromorphicExtensionOfTFractions2

Let $T(z)$ be a general limit periodic T-fraction of the form

$$T(z) = \cfrac{1}{1 + d_0 z - \cfrac{c_1 z}{1 + d_1 z - \cfrac{c_2 z}{1 + d_2 z - \cfrac{c_3 z}{\ddots}}}}$$

where c_n, $d_{n-1} \in \mathbb{C}$ are complex numbers with $c_n \neq 0$ for $n \geq 1$ and where $\lim_{n\to\infty} c_n = c \in \mathbb{C}$, $\lim_{n\to\infty} d_n = d \in \mathbb{C}$. Suppose further that the partial quotients of T satisfy

$$\sum_{j=1}^{\infty} \left(\left|c_j - c\right| + \left|d_j - d\right|\right) R^j < \infty$$

for some $R > 1$, let S denote the divergence line of T, and let $S(R)$ denote the boundary curve of the region into which a meromorphic extension of T across S exists. The following cases exhaust all possible values of $S(R)$ where, through⋱ out, $a = \left(R + R^{-1}\right)/2$ and $b = \left(R - R^{-1}\right)/2$.

1. If $d = 0$, then without loss of generality, $c = 1/4$. In this case,
$S(R) = \left\{z = r\, e^{i\psi} \,:\, r = r(\psi) = 2\,(a - \cos\psi)/b^2,\ 0 \leq \psi \leq 2\pi\right\}$

and $r'(\psi) > 0$. In this case, for R large, $S(R)$ is almost a circle of radius $4/R$ around 0; also, the endpoints of $S = [1, \infty$ are firs torder algebraic branch points for the extended meromorphic version of T presuming $T \not\equiv \infty$.

2. If $d = 1$, $c = |c|\, e^{i\vartheta}$, $\vartheta \in \mathbb{R}$, then $S(R)$ consists of two curves $S_{\pm}(R)$ defined as follows:

$S_{\pm}(R) = \left\{z = r_{\pm}\, e^{i\psi} \,:\, r_{\pm} = r_{\pm}(\psi) = P_{\pm}(\psi)/Q_{\pm}(\psi),\ \psi_1 \leq \psi \leq 2\pi - \psi_1\right\}$,

where $Q(\psi) = 2\,|c|\,(a - \cos(\psi - \vartheta)) > 0$, $p = a\,|c| + |c - 1| > 1$, $q = a\,|c| - |c - 1| > -1$, and

$P_{\pm}(\psi) = \sin^2\psi + \left(b\,|c| \pm ((p - \cos\psi)\,(q - \cos\psi))^{1/2}\right)^2$,

$\psi_1 \leq \psi \leq 2\pi - \psi_1$. Moreover:

(a) If $q \geq 1$, the $\psi_1 = 0$.

(b) If $q < 1$, ψ_1 denotes the unique solution of $\cos\psi_1 = q$, $0 < \psi_1 < \pi$.

(c) Always, $r_+(\psi) > r_-(\psi) > 0$ for $\psi_1 < \psi < 2\pi - \psi_1$.

(d) If $q < 1$, $0 < \psi_1 < \pi$, $r_+(\psi_1) = r_-(\psi_1) > 0$, $r_+(2\pi - \psi_1) = r_-(2\pi - \psi_1) > 0$.

(e) If $q = 1$, $\psi_1 = 0$, $r_+(0) = r_+(2\pi) = r_-(0) = r_-(2\pi) > 0$.

(f) If $q > 1$, $\psi_1 = 0$, $r_+(0) = r_+(2\pi) > r_-(0) = r_-(2\pi) > 0$. For large R, $S_{\pm}(R)$ are almost circles of radius $|c|\,R$, $(|c|\,R)^{-1}$, respectively. If $c = 1$, then $r_+(\psi) = R$, $r_-(\psi) = 1/R$ for $0 \leq \psi \leq 2\pi$.

TranscendentalCriterionForPartialQuotientsAndDenominators

Let $\alpha = [b_0; b_1, b_2, \ldots]$ be a continued fraction and suppose A_n / B_n denotes its nth convergent. Then α is transcendental if there exist functions $\epsilon : \mathbb{Z}^+ \to \mathbb{R}^+$, $k : \mathbb{Z}^+ \to \mathbb{Z}^+$ and constants δ, $c_1 \in \mathbb{R}$ for which (i) $b_{n+k(n)} \geq c_1 B_n^{\epsilon(n)}$ for infinitely many $n \in \mathbb{Z}^+$, and (ii) $\liminf_{n\to\infty} \left(\epsilon(n)/\delta - (1+\delta)^{k(n)-1}\right) > 0$.

TruncationBoundsForLimitPeriodicContinuedFractions1

Let ξ be a generalized continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{1}$$

and set

$$A = A_1$$

$$A_n = \sup_{m \geq n} \sqrt{|a_m|}$$

$$\alpha_n = \sqrt{a_m + \frac{1}{4}} - \frac{1}{2}$$

$$P_n = \frac{2 - A}{-\frac{2 A_n^2}{A} - A + 2}$$

$$\epsilon_n = P_n \sup_{m \geq n} |\alpha_m|$$

$$t_n = \frac{a_n}{z + 1}$$

and T_n be the composition of $t_1, \ldots, t_n$.

If ξ is a limit periodic continued fraction and

$$\lim_{n\to\infty} a_n = 0$$

and $A < 2/3$ and $\sup_{m \geq n} |\alpha_m| < (1 - A) P_1$, then ξ converges and

$$|T_n(\alpha_{n+1}) - T| < 2\, \epsilon_n \left(\prod_{m=1}^{n} \frac{A_m^2}{(1-\epsilon_m)^2}\right).$$

TruncationBoundsForLimitPeriodicContinuedFractions2

Let ξ be a generalized continued fraction

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{1}$$

where

$$\lim_{n\to\infty} a_n = a.$$

Define α_n by

$$\alpha_n\,(\alpha_n + 1) = a_n$$

and $|\alpha_n| < |\alpha_n + 1|$, then

$$\alpha\,(\alpha + 1) = a.$$

Set

$$t_n = \frac{a_n}{z + 1},$$

let T_n be the composition of $t_1, \ldots, t_n$, and let ξ be a limit periodic continued fraction. Then $\lim\inf(|-\alpha + \mu_n - 1|) > 0$ implies ξ converges and

$$\lim_{n\to\infty} (T_n(\mu_n)) = T.$$

TruncationErrorOfPositiveContinuedFraction

Let

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k}$$

be a continued fraction and $f_k = p_k / q_k$ the sequence of its convergents. Let a_k, $b_k > 0$ for all k. Then for any $m \geq 1$ the following holds:

$$0 < (-1)^n\,(f_{n+m} - f_n) \leq \frac{(-1)^{n+1}\,a_{n+1}(f_n - f_{n-1})}{b_n\,b_{n+1} + a_{n+1}\left(1 - a_n\,\frac{q_{n-2}}{q_n}\right)}.$$

TwoElementContinuedFractionRepresentationOfReals

Let α_1, α_2 be positive reals where $\alpha_1 < \alpha_2$, and set

$$\beta_1 = \frac{\sqrt{\alpha_1^2 \alpha_2^2 + 4 \alpha_1 \alpha_2} - \alpha_1 \alpha_2}{2 \alpha_2}$$

$$\beta_2 = \frac{\sqrt{\alpha_1^2 \alpha_2^2 + 4 \alpha_1 \alpha_2} - \alpha_1 \alpha_2}{2 \alpha_1}.$$

Given x is an irrational number where $0 < x < 1$, let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be the regular continued fraction of x. Let L_{A_2} be real numbers x where $b_n \in A_2 = \{\alpha_1, \alpha_2\}$. Then given $\alpha_1 \alpha_2 \leq 1/2$,

$L_{A_2} = [\beta_1, \beta_2]$.

UltraCloseApproximation

Let $\alpha \in (0, 1)$ be arbitrary. The rational number p/q is said to be an ultra-close approximation to α if among all rationals x/y with denominators $y \leq q$, p/q has the least ultra-distance to α, i.e., p/q is an ultra-close approximation to α if and only if

$$q \left| \frac{p}{q} - \alpha \right| = \min \left\{ y \left| \frac{x}{y} - \alpha \right| : \frac{x}{y} \in \mathbb{Q},\ y \leq q \right\}.$$

UltraDistance

Let $\alpha \in (0, 1)$ be arbitrary and let p/q be any rational number. The ultra-distance from p/q to α is defined to be $q\,|(p/q) - \alpha|$.

UltraDistancesAmongFareyPairsAndTheirMediants

Let a/b and c/d be a Farey pair with mediant $M = (a+c)/(b+d)$. Then the ultra-distance between a/b and M is the same as the ultra-distance between c/d and M.

UnboundedPeriodsForOddDegreeFamily

Let $d(X)$ be a polynomial, e be the exponent of $d(X)$, a be the leading coefficient set of $d(X)$ where X is an integer, $\sqrt{d(X)}$ be a quadratic irrational number, ξ be its regular continued fraction, and $l(X)$ be the regular continued fraction period of ξ. Given

$e \bmod 2 = 1 \lor \neg\, a = n^2$

then it follows that $l(X)$ is unbounded.

UnboundedPeriodsForSimpleQuadraticFamily

Let

$d(X) = r + X^2$

be a polynomial, X be an integer, r be an integer, $\sqrt{d(X)}$ be a quadratic irrational, ξ be its regular continued fraction, and $l(X)$ be the regular continued fraction period of ξ. Given $r \neq 0$, $r \neq -1$, $r \neq 1$, $r \neq 2$, $r \neq -2$, $r \neq 4$, and $r \neq -4$, it follows that $l(X)$ is unbounded.

UniformlyDistributedModuloOne

Let $E \subset [0, 1$, $\omega = \{x_n\}_{n=1}^{N}$ a sequence of real numbers and define $A(E; N; \omega)$ so that

$A(E; N; \omega) = \sharp\{n : 1 \leq n \leq N \text{ and } \mathrm{frac}(x_n) \in E\}$,

where $\sharp A$ denotes the number of elements of A for all sets A and $\mathrm{frac}(y)$ denotes the fractional part of the element y for all y. Then ω is said to be uniformly distributed modulo one if for every pair a, b with $0 \leq a < b \leq 1$, each interval $[a, b$ contains the "appropriate number of terms" in $A([a, b; N, \omega)$ as $N \to \infty$, i.e., if

$$\lim_{N\to\infty} \frac{A([a, b); N; \omega}{N} = b - a.$$

UnimodularMap

A homographic map $m : z \mapsto (a\,z + b)/(c\,z + d)$ is called unimodular if $a, b, c, d \in \mathbb{Z}[i]$ and $\det m = a\,d - b\,c \in \{\pm 1, \pm i\}$.

UniquenessOfCDuallyRegularExpansionsForIrrationals

Any irrational number $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ has precisely one C-dually regular continued fraction expansion.

UniquenessOfCRegularExpansionsForIrrationals

Any irrational number $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ has precisely one C-regular continued fraction expansion.

UniquenessOfIrrationalContinuedFractionExpansions

Let α_1 be an irrational number where $0 \leq \alpha_1 \leq 1$,

$$\xi_1 = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_1(n)}$$

be its regular continued fraction, α_2 be an irrational number where $0 \leq \alpha_2 \leq 1$, and

$$\xi_2 = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_2(n)}$$

be its regular continued fraction of α_2. Then given $\alpha_1 = \alpha_2$, it follows that $b_1(n) = b_2(n)$.

UniqueRegularChainRepresentationsOfCertainComplexNumbers

For any complex number $\xi \in \mathbb{C}$ which is not properly equivalent to a real number, there exists exactly one regular chain $c\,h\,\xi$ representing ξ.

VanVleckJensenTheorem

Let ξ be a regular continued fraction of the form

$$\xi = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}},$$

where each partial denominator b_k is an arbitrary complex number and let $w_n = [0; b_1, b_2, \ldots, b_n]$ denote the nth convergent of ξ. Suppose further that $\mathrm{Re}(b_n) > 0$ for all n and that, for $\theta < \pi/2$ arbitrary, $|\arg(b_n)| < \theta$. Then:

The sequences $\{w_{2n}\}$ and $\{w_{2n+1}\}$ of even and odd convergents of ξ, respectively, converge.

The sequence $\{w_n\}$ converges if and only if $\sum_{n=1}^{\infty} |b_n| = \infty$.

For all $m \geq n$, $|w_m - w_{n-1}| \leq 1/d_n$ for $d_n \geq \kappa \cos(\theta) \ln\left(1 + \lambda \cos(\theta) \sum_{k=1}^{n} |b_k|\right)$. Here, $\kappa = \mathrm{Re}(b_1)/(2 + \mathrm{Re}(b_1))$ and $\lambda = (\mathrm{Re}(b_1))^2 \min\left\{1, 1 \middle/ |b_1|^2\right\}$.

VanVleckTheorem

Let

$$\xi = \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_k}$$

be a continued fraction with $b_1 \neq 0$ and all the $b_k \in \mathbb{C}$ with $\mathrm{Re}(b_k) > 0 \vee b_k = 0$. Then ξ converges if and only if

$$\sum_{k=1}^{\infty} |b_k| = \infty.$$

VanVleckTheoremOnConvergenceOfRegularCFractions

Let ξ be a regular C-fraction,

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{a_n z}{1},$$

f be a meromorphic function, Γ be $\{-t/(4a) : t \geq 1\}$,

$D = \mathbb{C} - \Gamma$

be a domain, V be the poles for f in D, and K be any complex compact set in D disjoint from V and Γ. Then given

$$\lim_{n \to \infty} a_n = a,$$

there is a meromorphic function f such that $\forall_K \, \xi$ converges uniformly on K to f.

VeryWellApproximableNumbersConvergentDenominatorsDivergeInLogarithmicMean

Let

$$\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{b_n}$$

be a regular continued fraction, B_n be the convergent denominator of ξ, ϵ be a positive real, and $S(\epsilon)$ be the natural numbers n where $b_{n+1} > B_n^{\epsilon}$. Then the existence of an ϵ such that $S(\epsilon)$ is finite if and only if ξ is well approximable, and if ξ is very well approximable, then $\lim_{n\to\infty} \ln(B_n)/n$ does not converge.

VincentTheorem

Given a polynomial equation with rational coefficients that does not have multiple roots, making successive transformations of the form

$$x = b_1 + \frac{1}{x'},\ x' = b_2 + \frac{1}{x''},\ x'' = b_3 + \frac{1}{x'''},\ \ldots$$

where $b_1, b_2, \ldots$ are any positive numbers $b_i \geq 1$, the resulting transformed equation has either zero or one sign variations.

If there are zero sign variations, the polynomial equation has no root.

If there is one sign variation, the polynomial equation has a single positive real root represented by the continued fraction

$$b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \frac{1}{\ddots}}}.$$

WaadelandTailTheorem

A sequence $\{g^{(n)}\}_{n=0}^{\infty}$ of nonzero complex numbers satisfying $g^{(k)} \neq -1$ for $k = 1, 2, 3, \ldots$ is the sequence of right tails for some convergent continued fraction

$$\xi = \underset{m=1}{\overset{\infty}{\mathrm{K}}} \frac{b_m}{1},$$

$b_k \in \mathbb{C} \setminus \{0\}$, $k = 1, 2, 3, \ldots$, if and only if

$$1 + \kappa_1 + \kappa_1 \kappa_2 + \kappa_1 \kappa_2 \kappa_3 + \cdots = \infty,$$

where

$$\kappa_n = \frac{-1 + g^{(n)}}{g^{(n)}}$$

for $n = 1, 2, 3, \ldots$. When this result does hold, the elements b_k of ξ necessarily have the form

$$b_{k+1} = g^{(k)} \left(1 + g^{(k+1)}\right)$$

for $k = 0, 1, 2, \ldots$.

WallTransformation

The phrase "Wall transformation" is an unofficial term referring to a certain transform of complex-valued functions studied by H.S. Wall, among others, and is notable for its end result, namely the expression of a complex-valued function f as an equivalent continued fraction. Not to be confused with the closely-related Schur algorithm for complex-functions, the Wall transform describes more so the underlying continued fraction theory of the elements used by Schur in his algorithm. A more precise version of this distinction is as follows. Given a function $f = f_0 : \Omega \to \mathbb{C}$ where $\Omega \subset \mathbb{C}$ is a region, Schur's algorithm determines a sequence $\{f_n\}_{n=1}^{\infty}$ of complex-valued functions for which

$$f_n(z) = \frac{z\, f_{n+1}(z) + f_n(0)}{1 + \overline{f_n(0)}\, z\, f_{n+1}(z)} = f_n(0) + \frac{\left(1 - |f_n(0)|^2\right) z}{\overline{f_n(0)}\, z + 1 / f_{n+1}(z)}.$$

Substituting the resulting expressions in terms of lower-indexed terms, one obtains for f the so-called Wall continued fraction ξ_f of the form

$$\xi_f = a_0 + \frac{\left(1 - |a_0|^2\right) z}{\overline{a_0}\, z + \frac{1}{a_1 + \frac{\left(1 - |a_1|^2\right) z}{\overline{a_1}\, z + \cdots}}}$$

where $a_n = f_n(0)$ for $n = 1, 2, 3, \ldots$. By way of the maximum principle, the process stops if $|a_n| = 1$ for some n and continues ad infinitum otherwise. The Wall transformation, then, is the collection $\{\tau_n\}_{n=0}^{\infty}$ of Möbius transformations where for each n, $\tau_n(w)$ is of the form

$$\tau_n(w) = \frac{z\, w + a_n}{1 + \overline{a_n}\, w}$$

and is related to the aforementioned algorithm of Schur by the identity

$$f(z) = \tau_0 \circ \tau_1 \circ \cdots \circ \tau_n (f_{n+1}),$$

where n is either the index for which $|a_n| = 1$ or $n = \infty$ otherwise. Analogous to the typical recurrence notation for continued fraction convergents, the above identity for f in terms of τ_k leads to the expression

$$\tau_0 \circ \tau_1 \circ \cdots \circ \tau_n (w) = \frac{A_n + z\, B_n^*\, w}{B_n + z\, A_n^*\, w}$$

for all $w \in \mathbb{C}$, where $\{A_n\}$, $\{B_n\}$ are collections of polynomials (called Wall polynomials) and where $p_n^*(z) = z^n\, \overline{p_n(1/\overline{z})}$ for any polynomial p_n. Here, by definition, $B_0 = B_0^* = 1$, $A_0 = a_0$, and $A_0^* = \overline{a_0}$. Using this construction, one can immediately prove analogous versions of the determinant continued fraction identity, along with a wide array of identities concerning analytic functions of the unit ball, Blaschke products, and orthogonal polynomials.

WilliamsConjecture

The period length $p(d)$ of a regular continued fraction expansion of $\sqrt{d}$ for positive integer d should, under the extended Riemann hypothesis, be bounded above by $c\sqrt{d}\ \ln(\ln(d))$ for a suitable c:

$$\frac{p(d)}{\begin{cases}\sqrt{d}\ \ln(\ln(d)) & \text{for } d = 1 \pmod 8\\ \sqrt{d}\ \ln(\ln(4\,d)) & \text{for } d \neq 1 \pmod 8\end{cases}} < \tilde{c} + o(1),$$

where $\tilde{c} = 3.7012$.

Possibly, $\tilde{c} = 12\exp(\gamma)\ln(2)/\pi^2 \approx 1.501$.

WorpitzkyTheorem

Let $\xi = \underset{n=1}{\overset{\infty}{\mathrm{K}}}\, a_n/1$ be a generalized continued fraction with partial numerators a_n satisfying $0 < |a_n| \leq 1/4$ for all $n \geq 1$. Then

1) ξ converges absolutely for some value of ξ with $0 < |\xi| \leq 1/2$

2) $0 < |S_n(w)| \leq 1/2$ for all $n \in \mathbb{Z}^+$ and $|w| \leq 1/2$, where $S_n(w)$ is the nth approximant function.

ZajtaPandikovContinuedFractionToPowerSeriesConversion

The continued fraction

$$\xi = 0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{\begin{cases}1 & \text{for } k = 1\\ -z_k\, t & \text{for } k \neq 1\end{cases}}{1}$$

has the following equivalent power series representation

$$\xi = 1 + \sum_{k=1}^{\infty} F_n(z_1, z_2, \ldots, z_n)\, t^n$$

where the coefficients $F_n(z_1, z_2, \ldots, z_n)$ are

$$F_n(z_1, z_2, \ldots, z_n) = \sum_{p(n)} \left(\sum_{k=2}^{n} \binom{c_{k-1} + c_k - 1}{c_k} {z_k}^{c_k} \right)$$

where the outer sum extends over all unordered partitions $p(n)$ of the integer n into n nonnegative parts c_k: $\sum_{k=1}^{n} c_k = n$.

ZarembaConjecture

Let $\mathfrak{R}_A$ be the set of all finite continued fractions with all partial denominators bounded by an integer $A > 0$:

$$\mathfrak{R}_A = \left\{ \frac{b}{d} = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k} : \underset{1 \le k \le N}{\forall} 1 \le b_k \le A \bigwedge N \in \mathbb{Z}^* \bigwedge N < \infty \right\}.$$

Let $\mathfrak{D}_A$ be the set of all denominators occurring in $\mathfrak{R}_A$:

$$\mathfrak{D}_A = \left\{ d : \exists_b \gcd(b, d) = 1 \bigwedge \frac{b}{d} \in \mathfrak{R}_A \right\}.$$

Then for sufficiently large A, $\mathfrak{D}_A = \mathbb{Z}^+$ holds.

ZarembaConjectureForLargeA

Let $\mathfrak{R}_A$ be the set of all finite continued fractions with all partial denominators bounded by an integer $A > 0$:

$$\mathfrak{R}_A = \left\{ \frac{b}{d} = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k} : \underset{1 \le k \le N}{\forall} 1 \le b_k \le A \bigwedge N \in \mathbb{Z}^* \bigwedge N < \infty \right\}.$$

Let $\mathfrak{D}_A$ be the set of all denominators occurring in $\mathfrak{R}_A$:

$$\mathfrak{D}_A = \left\{ d : \exists_b \gcd(b, d) = 1 \bigwedge \frac{b}{d} \in \mathfrak{R}_A \right\}.$$

Then for sufficiently large A, and $N \in \mathbb{Z}^*$,

$\mathrm{card}(\mathfrak{D}_A \bigcap [1, N]) = N(1 + o(1))$.

ZarembaConjectureForSmallPowers

Let $\mathfrak{R}_A$ be the set of all finite continued fractions with all partial denominators bounded by an integer $A > 0$:

$$\mathfrak{R}_A = \left\{ \frac{b}{d} = \underset{k=1}{\overset{N}{\mathrm{K}}} \frac{1}{b_k} : \underset{1 \le k \le N}{\forall} 1 \le b_k \le A \bigwedge N \in \mathbb{Z}^* \bigwedge N < \infty \right\}.$$

Let $\mathfrak{D}_A$ be the set of all denominators occurring in $\mathfrak{R}_A$:

$$\mathfrak{D}_A = \left\{ d : \exists_b \gcd(b, d) = 1 \bigwedge \frac{b}{d} \in \mathfrak{R}_A \right\}.$$

Then all powers of 2 and all powers of 3 are in $\mathfrak{D}_3$.