Appendix F: Examples of existing semantic representations of mathematics

1. The central limit theorem as represented in Isabelle [12]:

```
theorem (in prob_space) central_limit_theorem:
  fixes
    X :: "nat ⇒ 'a ⇒ real" and
    μ :: "real measure" and
    σ :: real and
    S :: "nat ⇒ 'a ⇒ real"
  assumes
    X_indep: "indep_vars (λi. borel) X UNIV" and
    X_integrable: "⋀n. integrable M (X n)" and
    X_mean_0: "⋀n. expectation (X n) = 0" and
    σ_pos: "σ > 0" and
    X_square_integrable: "⋀n. integrable M (λx. (X n x)²)" and
    X_variance: "⋀n. variance (X n) = σ²" and
    X_distrib: "⋀n. distr M borel (X n) = μ"
  defines
    "S n ≡ λx. ∑i<n. X i x"
  shows
    "weak_conv_m (λn. distr M borel (λx. S n x / sqrt (n * σ²)))
        (density lborel standard_normal_density)"
```

2. The definition of a Möbius transformation in Coq [28]:

```
Definition Mobius (R : ringType) (p : {poly R}) (a b : R) :=
  reciprocal_pol ((p \shift a) \scale (b - a)) \shift 1.

Definition reciprocal_pol (R : ringType) (p : {poly R}) :=
  \poly_(i < size p) p`_(size p - i.+1).

Definition scaleX_poly (R : ringType) (c : R) (p : {poly R}) :=
  p \Po ('X * c).

Definition shift_poly (R : ringType) (c : R) (p : {poly R}) :=
  p \Po ('X + c).
```

3. The definition of semigroup as a structure predicate in Coq [52]:

```
Record SemiGroup (G: Type) (e: relation G) (op: G → G → G): Prop :=
  { sg_setoid: Equivalence e
  ; sg_ass: Associative op
  ; sg proper: Proper (e ⇒ e ⇒ e) op }.
```

4. A simple theorem in Mizar [94]:

```
for a,b,c st
 a^2 + b^2 = c^2 & a,b are_relative_prime & a is odd holds
  ex m,n st m <= n & a = n^2 - m^2 & b = 2*m*n & c = n^2 + m^2;
```

5. A much more sophisticated theorem in Mizar [100]:

```
::$N Baire Category Theorem for Continuous
::    Lattices
theorem Th39: :: Theorem 3.43.7
  for L being lower-bounded continuous LATTICE
  for D being non empty countable dense
          Subset of L,
      u being Element of L st u <> Bottom L
   ex p being irreducible Element of L st
   p <> Top L & not p in uparrow ({u} "/\" D)
```

6. A proof-oriented theorem from about prime numbers and divisibility (If p is prime and $p|a*b$, then $p|a$ or $p|b$) in MathAbs [32]:

Theorem. let a, b : NoType let p : Prime
assume $p \mid a * b$ show $p \mid a$ or $p \mid b$ •

7. The forward regular continued fraction expansion algorithm in the Wolfram Language [13]:

```
RegularContinuedExpansion[x_ /; Element[x, ℝ], n_] :=
   With[{b0 = Floor[x],
          τ = Function[ξ, (1/ξ - ⌊1/ξ⌋)]},
  Join[{b0}, Floor[1 / Most[NestWhileList[τ, x - b0, Function[ξ, ξ ≠ 0], 1, n]]]]
  ]
```

8. The definition of a vector in AGDA:

Definition 2.1. data Vec (A : Set) : ℕ → Set **where**
  [] : Vec A zero
  _::_ : ∀ {n} (x : A) (xs : Vec A n) → Vec A (suc n)

9. A possible representation of the statement "Any compact subspace C of a Hausdorff space T is closed in T," as suggested in [98]:

$$\forall \boldsymbol{T}.(topological\_space(T) \wedge hausdorff(T) \Rightarrow$$
$$(\forall \boldsymbol{C}.(subspace(C,T) \wedge compact(C) \Rightarrow closed(C,T)))).$$

10. An axiom in Russel [101]:

```
axiom ax-mp (var ph : wff , var ps : wff )
{
   hyp 1 : wff = |- ph ;
   hyp 2 : wff = |- ( ph -> ps ) ;
   ------------------------------
   prop : wff = |- ps ;
}
```

11. Moessner's Theorem in Nuprl [103]:

```
∀[x,y:Atom].
   ∀[n:ℕ]. ∀[k:ℕ+].
      (Moessner(ℤ-rng;x;y;1;λi.if (i =z 0) then 0
                     if (i =z 1) then n else 0 fi ;k)[bag-rep(n;x)]
       = k^n)
   supposing ¬(x = y)
```

12. The representation as an ACL2 fragment in a recent paper [21] of

$$\forall_n\ n > M \Rightarrow \left|\sum_{i=0}^{n} a_i - L < \epsilon\right|$$

is:

```
(defun-sk All-n-abs-sumSer1-upto-n-L<eps (L eps M)
  (forall n (implies (and (standardp n)
                          (integerp n)
                          (> n M))
                     (< (abs (- (sumSer1-upto-n n) L))
                        eps))))
```